"Core Competency" (June 2022) Qualifying Exam (06/07/2022)
Time Limit: 4 hours

Name (Print): ____________________

Signature: ____________________

This exam contains 8 problems. Answer **all** of them. Point values are in parentheses. You **must show your work** to get credit for your solutions — correct answers without work will not be awarded points. Please use separate sheet(s) for each question.

No calculators will be allowed in the exam. This is a closed notes/book exam; no cheat sheets are allowed.

1	12 pts	
2	12 pts	
3	12 pts	
4	16 pts	
5	16 pts	
6	10 pts	
7	12 pts	
8	10 pts	
TOTAL	100 pts	

1. (12 points) (6+6) Assume that observations X_1 and X_2 are jointly normally distributed with $E(X_1) = E(X_2) = \theta$, $Var(X_1) = 1$ and $Var(X_2) = 2$, where θ is the parameter of interest.

   (i) Suppose that $Cov(X_1, X_2) = 0$. Among all unbiased estimators, find one which achieves the minimum variance. Justify your answer.

   (ii) Suppose that $Cov(X_1, X_2) = 1$. Find an unbiased estimator with minimum variance. Justify your answer. Hint: k-variate normal density has the following form

$$f(x) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp(-(x-\mu)^T \Sigma^{-1}(x-\mu)/2).$$

2. (12 points) (6+6) Let a positive random variable T have a hazard rate function of the form

$$\lambda(t) = \gamma t^{\gamma - 1}, \quad \gamma > 0.$$

   (i) Find its probability density function f and its median. (Hint: Recall that $\lambda(t) = f(t)/(1 - F(t))$, where F is the cumulative distribution function.)

   (ii) Find its mean for the cases of $\gamma = 1$ and $\gamma = 2$.

3. (12 points) (6+6) Let Y be a standard d-dimensional normal random vector and let P be a $d \times d$ symmetric and idempotent matrix, i.e. $P = P^T$ and $P^2 = P$.

   (i) Give all the possible values of the eigenvalues of P.

   (ii) Show that $Y^T P Y \sim \chi^2(k)$ where $k = \text{rank}(P)$.

4. (16 points) (4+4+4+4) A researcher conducts a completely randomized experiment to test a drug, with n subjects $i = 1, ...n$, n_{0+} of which are assigned to a control group, n_{1+} to a treatment group: let $X_i = 0$ if subject i is in the control group, $X_i = 1$ if i is in the treatment group. Let $Y_i = 0$ if i does not improve, $Y_i = 1$ if i improves. The researcher observes n_{00} subjects in the control group and n_{10} subjects in the treatment group with $Y = 0$, n_{01} subjects in the control group and n_{11} subjects in the treatment group with $Y = 1$. You may assume that both n_{0+} and n_{1+} are "large", that subjects in the control group are independent and identically distributed (iid) with $\Pr(Y = 1 \mid X = 0) = \pi_0$, subjects in the treatment group are iid with $\Pr(Y = 1 \mid X = 1) = \pi_1$. Let T_0 denote the number of successes in the control group, T_1 the number of successes in the treatment group.

   (i) Write out the probability distribution $\Pr(T_0 = n_{01}, T_1 = n_{11})$.

   (ii) The researcher is interested in testing $H_0 : \pi_0 = \pi_1$ vs. $H_A : \pi_0 \neq \pi_1$. As a test statistic, she uses: $\frac{p_1 - p_0}{(\frac{p_1(1-p_1)}{n_{1+}} + \frac{p_0(1-p_0)}{n_{0+}})^{1/2}}$ where $p_0 = n_{01}/n_{0+}$, $p_1 = n_{11}/n_{1+}$ and compares this to the normal distribution with mean 0 and variance 1. Justify the use of this statistic.

(iii) Construct the likelihood ratio test for H_0 vs. H_A above.

(iv) The researcher is also interested in estimating the relative risk RR= $\Pr(Y = 1 \mid X = 1)/\Pr(Y = 1) \mid X = 0)$. What is the maximum likelihood estimator of RR?

5. (16 points) (4+4+4+4) Suppose that observations Y_i, $i = 1, 2, 3$, follow independent Poisson distributions with parameters λ_i.

   (i) What is the distribution of $Y_1 + Y_2 + Y_3$? Justify.

   (ii) The researcher believes $\lambda_i = i \times \lambda$ for all $i = 1, 2, 3$. How might you test $H_0 : \lambda_i = i \times \lambda$ for all i vs. $H_A : \lambda_i \neq i \times \lambda$ for some i?

   (iii) Presuming the researcher's belief is correct, obtain the maximum likelihood estimator of λ.

   (iv) Again, presuming the researcher's belief is correct, obtain the likelihood ratio test for $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda \neq \lambda_0$.

6. (10 points) Suppose that $\widehat{\theta}$ has the p-dimensional normal distribution with expectation the vector all of whose components are zero and positive definite variance-covariance matrix Σ. Let $c(\widehat{\theta})$ be the value of c that maximizes

$$t = \frac{c^T \widehat{\theta}}{\sqrt{c^T \Sigma c}}.$$

   Find the distribution of

$$t_{max} = \frac{c(\widehat{\theta})^T \widehat{\theta}}{\sqrt{c(\widehat{\theta})^T \Sigma c(\widehat{\theta})}}.$$

7. (12 points) Suppose that Y_{i1}, Y_{i2}, independent pairs of Bernoulli observations with expectations, respectively,

$$\frac{e^{\alpha_i + \beta}}{1 + e^{\alpha_i + \beta}} \text{ and } \frac{e^{\alpha_i}}{1 + e^{\alpha_i}},$$

   i from 1 to n, where the α_i and β are unspecified parameters. Let $\widehat{\beta}$ be the maximum likelihood estimate of β. What is the limiting expectation of $\widehat{\beta}$ when the true value of β is 0.5 and the true values of α_i are all equal to 0? Hint: for each i, think about the four possibilities for (Y_{i1}, Y_{i2}) separately.

8. (10 points) Suppose we have a normal random sample $X_1, \ldots, X_n$ such that $\mathbb{E}[X_i] = \mu$, and that we conduct a t-test with level α of the hypothesis $H_0 : \mu = \mu_0$. Show that the t-test achieves an asymptotic level α when we have n independent $X_i \sim N(\mu, \sigma_i^2)$ if $0 < m \leq \sigma_i \leq M < \infty$ for all $i = 1, \ldots, n$, where $m < M$ are positive constants that do not depend on n.