1. (10 points) (4 + 6) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, where $\lambda, \mu > 0$ and assume that X and Y are independent.

(i) Find the conditional distribution of X given that $X + Y = n$.

(ii) Use the above, or otherwise, to test the hypothesis (at level $\alpha \in (0, 1)$)

$$H_0 : \lambda = \mu \qquad \text{versus} \qquad \lambda > \mu.$$

2. (8 points) Let $Z_1, \dots, Z_n$ be i.i.d. random variables with density f. Suppose that (i) $\mathbb{P}(Z_i > 0) = 1$, and (ii) f is continuous on $[0, \epsilon)$, for some $\epsilon > 0$. Let $\lambda := f(0)$. Let

$$X_n = n \min\{Z_1, \dots, Z_n\}.$$

Show that X_n converges in distribution, and find the limiting distribution.

3. (12 points) (2+2+5) n_1 people are given treatment 1 and n_2 people are given treatment 2. Let X_1 be the number of people on treatment 1 who respond favorably to the treatment and let X_2 be the number of people on treatment 2 who respond favorably. Assume that $X_1 \sim \text{Binomial}(n_1, p_1)$, and $X_2 \sim \text{Binomial}(n_2, p_2)$. Let $\psi = p_1 - p_2$.

(i) Find the maximum likelihood estimator $\hat{\psi}$ of ψ.

(ii) Find the Fisher information matrix $I(p_1, p_2)$.

(iii) Use the delta method to find the asymptotic standard error of $\hat{\psi}$.

4. (15 points) (4 + 3 + 4 + 4) We model the lifetime of a device as a random variable $T \geq 0$ with c.d.f. $F(t)$ and density $f(t)$. Suppose that $f(t)$ is continuous for $t \geq 0$ and define the intensity of failure as

$$\lambda(t) = \lim_{h \downarrow 0} \frac{P[t \leq T \leq t + h | T \geq t]}{h} \qquad \text{for } t \geq 0.$$

(a) Express $\lambda(t)$ through $f(t)$ and $F(t)$.

(b) Compute the intensity of failure when $T \sim \text{Exp}(\alpha)$, $\alpha > 0$.

(c) Show that $F(t) = 1 - \exp\{-\int_0^t \lambda(s)\, ds\}$ for $t \geq 0$.

(d) Determine $F(t)$ and $f(t)$ in the case that $\lambda(t) = \alpha t^\gamma$ for some $\alpha > 0$ and $\gamma > 0$.

5. (8 points) You are working as a TA in the help room for a duration t. The number of students arriving during that period is Poisson distributed with parameter $t\lambda$. For each student, the time T to answer their questions is exponentially distributed with parameter α and this time is independent of all other students. Prove that the distribution of the number X of students that arrive while you are busy with one fixed (randomly chosen) student is geometric with some parameter p and determine p in terms of α and λ.
Hint: The formula $\int_0^\infty s^k e^{-s}\, ds = k!$ for $k = 0, 1, 2, \dots$ can be used without proof.

6. (10 points) (4 + 6) Denote by $\hat{\zeta}_n$ the MLE of $\zeta = p(1 - p)$ based on n i.i.d. samples from $\text{Binomial}(1, p)$. Denote by p_0 the true value of p.

(a) If $p_0 \neq 1/2$, find the limiting (non-degenerate) distribution of $\hat{\zeta}_n$, with proper normalization.

(b) Derive the asymptotic distribution of $\hat{\zeta}_n$, when $p_0 = 1/2$.

7. (10 points) (5 + 5) Suppose you have n red balls and one blue ball. We will do two experiments.

(a) In the first experiment, you first drop the n red balls uniformly on the interval $[0, 1]$, independent of each other. Having done this, now you drop the blue ball uniformly in the interval $[0, 1]$, independent of previous ball drops. Let X denote the number of red balls to the left of the blue ball. Find $\mathbb{P}(X = k)$, for $k = 0, \dots, n$.

(b) In the second experiment, you drop all the $(n + 1)$ balls uniformly on $[0, 1]$, independent of each other. Let Y denote the number of red balls to the left of the blue ball as before. Find $\mathbb{P}(Y = k)$, for $k = 0, \dots, n$.

8. (15 points) (7 + 8) Suppose you have a quadratic form $\mathbf{X}_n^T A_n \mathbf{X}_n$, where $\mathbf{X}_n \sim N_n(\mathbf{0}_{n\times 1}, \mathbf{I}_{n\times n})$, and A_n is a symmetric $n \times n$ matrix with 0 on the diagonal. Let $(\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ denote the eigenvalues of A_n, and let $\|\lambda\|_{2,n} := \sqrt{\sum_{i=1}^n \lambda_{i,n}^2}$ denote the ℓ_2-norm of the eigenvalues.

(a) If $\lim_{n\to\infty} \frac{\max_{i=1,\ldots,n} |\lambda_{i,n}|}{\|\lambda\|_{2,n}} = 0$, show that $T_n := \frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \xrightarrow{\text{d}} N(0,1)$ as $n \to \infty$.
[Hint: You may use Lyapunov's[1] CLT. Note that the trace of a square matrix is the sum of its eigenvalues.]

(b) If

$$\lim_{n\to\infty} \frac{\lambda_{1,n}}{\|\lambda\|_{2,n}} = 1,$$

show that $T_n \xrightarrow{\text{d}} \chi_1^2 - 1$.

9. (8 points) (3 + 5) Let $Y_n = \prod_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are i.i.d. nonnegative non-degenerate random variables with mean $\mathbb{E}(X_i) = 1$. Prove that $Y_n \xrightarrow{\text{P}} 0$ as $n \to \infty$ when: (i) $\mathbb{P}(X_1 = 0) > 0$, and (ii) $\mathbb{P}(X_1 = 0) = 0$.

10. (4 points) Let $f_{X,Y}(x,y)$ be a bivariate density and let $(X_1, Y_1), \ldots, (X_N, Y_N)$ be i.i.d. $f_{X,Y}$. Let $w(\cdot)$ be an arbitrary probability density function. Let

$$\hat{f}_X(x) = \frac{1}{N} \sum_{i=1}^N \frac{f_{X,Y}(x, Y_i) w(X_i)}{f_{X,Y}(X_i, X_i)}.$$

Show that, for any $x \in \mathbb{R}$, $\hat{f}_X(x) \xrightarrow{\text{P}} f_X(x)$, where f_X is the marginal density of X_1.

[1]**Lyapunov's CLT**: Suppose that $\{Z_1, Z_2, \ldots\}$ is a sequence of independent random variables such that Z_i has finite expected value μ_i and variance σ_i^2. Define $s_n^2 := \sum_{i=1}^n \sigma_i^2$. If $\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}[|Z_i - \mu_i|^3] = 0$ is satisfied, then $\frac{1}{s_n} \sum_{i=1}^n (Z_i - \mu_i) \xrightarrow{\text{d}} N(0,1)$.