"Core Competency" (**May 2020**) **Name (Print):** ____________________
Qualifying Exam (05/27/2020)
Time Limit: 4 hours **Signature:** ____________________

This exam contains 9 problems. Answer **all** of them. Point values are in parentheses. You **must show your work** to get credit for your solutions — correct answers without work will not be awarded points. Please use separate sheet(s) for each question.

No calculators will be allowed in the exam. This is a closed notes/book exam; no cheat sheets are allowed.

1	8 pts	
2	10 pts	
3	12 pts	
4	10 pts	
5	12 pts	
6	12 pts	
7	12 pts	
8	12 pts	
9	12 pts	
TOTAL	100 pts	

1. (8 points) Suppose we have a random variable $\xi \sim \text{Uniform}(0,1)$. Suppose that conditioning on ξ, we have i.i.d. Bernoulli(ξ) random variables $X_1, X_2, \ldots, X_n, X_{n+1}$, i.e. $P(X_i = 1|\xi) = 1 - P(X = 0|\xi) = \xi$. Calculate

$$P(X_{n+1} = 1 \mid X_1, X_2, \ldots, X_n).$$

2. (10 points) (5+5) Let $X_1, X_2, \ldots, X_n$ denote n independent and identically distributed observations from Uniform$(0,1)$. We order these observations according to their distance from $x = 0.75$ and call the ordered ones $X^x_{(1)}, X^x_{(2)}, \ldots, X^x_{(n)}$. Note that $X^x_{(1)}$ and $X^x_{(n)}$ are, respectively, the closest and farthest observations from $x = 0.75$.

   (i) Prove that $X^x_{(1)}$ converges to 0.75 in probability.

   (ii) What does $X^x_{(n)}$ converge to in probability? Prove your answer.

3. (12 points) (4+4+4) Suppose $N, \{X_i\}_{i\geq 1}$ are i.i.d. Poisson random variables with mean 1. Let $T = \sum_{i=1}^{N} X_i$.

   (i) Compute expectation $E(T)$.

   (ii) Compute variance $Var(T)$.

   (ii) Find $\mathbb{P}(T = 1)$ as explicitly as possible.

4. (10 points) (5+5) Let Z_1, Z_2, Z_3 be i.i.d. N(0,1) random variables. Let $R = \sqrt{Z_1^2 + Z_2^2 + Z_3^2}$.

   (i) Find the distribution of R and write down its density function.

   (ii) Suppose that we have two independent random variables $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$, where $\alpha, \beta, \lambda > 0$. Let

$$U = X + Y \qquad \text{and} \qquad V = \frac{X}{X+Y}.$$

   Find the joint density (p.d.f.) of (U, V) and identify the joint distribution (c.d.f.). **Hint**: density function of Gamma$(\alpha, \lambda) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x}/\Gamma(\lambda)$.

5. (12 points) (4+4+4) Suppose that $X_1, \ldots, X_n$ are i.i.d. observations from the exponential distribution with parameter λ (recall that $\mathbb{E}(X_1) = \lambda^{-1}$). Consider the following testing problem:

$$H_0 : \lambda = \lambda_0 \qquad \text{versus} \qquad H_1 : \lambda = \lambda_1,$$

   where $0 < \lambda_1 < \lambda_0$. Let $f_0(X_1, \ldots, X_n)$ be the likelihood of the data under H_0 and $f_1(X_1, \ldots, X_n)$ that under H_1.

   (i) Show that $\log \frac{f_1(X_1,\ldots,X_n)}{f_0(X_1,\ldots,X_n)}$ is an increasing function of $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$.

(ii) Suppose that $c_{\alpha,n}$ is such that $\mathbb{P}_{\lambda_0}(\bar{X}_n \geq c_{\alpha,n}) = \alpha$, for $\alpha \in (0,1)$. Relate $c_{\alpha,n}$ to $q_k(\beta)$ — the β'th quantile of the χ^2_k distribution (for some k).

(iii) How would you test the hypothesis

$$H_0 : \lambda = \lambda_0 \qquad \text{versus} \qquad H_1 : \lambda < \lambda_0?$$

Derive an expression for the power function of the test.

6. (12 points) (4+4+4) Suppose that we have single observation from X from the exponential distribution with parameter λ. Define $T(X) = I(X > 1)$, where I is the indicator function. Set $\psi(\lambda) := e^{-\lambda}$.

   (i) Show that $T(X)$ is unbiased for $\psi(\lambda)$.

   (ii) Find the (Fisher) information bound for unbiased estimators of $\psi(\lambda)$.

   (iii) Show that the variance of $T(X)$ is strictly larger than the information bound.

7. (12 points) (4+4+4) Consider the random variable $X = \mu + \sigma Z$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and Z is a random variable with a density f. Suppose that μ and σ are unknown parameters and that the density f is known (completely specified). We have a random i.i.d sample $X_1, \ldots, X_n$ with the same distribution as X.

   (i) Propose unbiased estimators, $\hat{\mu}$ and $\hat{\sigma}^2$, of μ and σ^2.

   (ii) Does the joint distribution of $(X_i - \hat{\mu})/\hat{\sigma}$ $(i = 1, \ldots, n)$ depend on μ and σ? Explain your answer.

   (iii) For a given level $\alpha \in (0,\ 1)$, describe a way to construct a confidence interval for μ with exact coverage probability $1 - \alpha$.

8. (12 points) (4+4+4) Let $X_1, \ldots, X_n$ be an i.i.d. Bernoulli(p) random sample, i.e. $P(X_i = 1) = 1 - P(X_i = 0) = p$, $p \in (0,1)$. Further let $\theta = Var(X_i)$.

   (i) Find $\hat{\theta}$, the maximum likelihood estimator of θ.

   (ii) Show that $\hat{\theta}$ is asymptotically normal when $p \neq \frac{1}{2}$ and give the asymptotic variance.

   (iii) When $p = \frac{1}{2}$, derive a non-degenerated asymptotic distribution of $\hat{\theta}$ with an appropriate normalization. *Hint: try relating $\hat{\theta}$ to the statistic $(\overline{X}_n - 1/2)^2$*

9. (12 points) (4+4+4) Let $X_1, \ldots, X_{2n}$ be an i.i.d. random sample with common pdf $f(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}$ for $x > 0$. Consider the three estimators $\hat{\lambda}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\lambda}_2 = \frac{1}{n}\sum_{i=n+1}^{2n} X_i$ and $\hat{\lambda} = \frac{1}{2n}\sum_{i=1}^{2n} X_i$.

(i) Show that $T_1 = \hat{\lambda}_1 \hat{\lambda}_2$ is an unbiased and consistent estimator of λ^2.

(ii) Show that $T_2 = \hat{\lambda}^2$ is consistent for λ^2, but not unbiased.

(iii) Derive the asymptotic distribution of the estimators T_1 and T_2. Which one is more efficient asymptotically?