"Core Competency" (**May 2021**) **Name (Print):** ____________________
Qualifying Exam (05/24/2021)
Time Limit: 4 hours **Signature:** ____________________

This exam contains 8 problems. Answer **all** of them. Point values are in parentheses. You **must show your work** to get credit for your solutions — correct answers without work will not be awarded points. Please use separate sheet(s) for each question.

No calculators will be allowed in the exam. This is a closed notes/book exam; no cheat sheets are allowed.

1	14 pts	
2	14 pts	
3	10 pts	
4	16 pts	
5	12 pts	
6	12 pts	
7	12 pts	
8	10 pts	
TOTAL	100 pts	

1. (14 points) (5+5+4) Let $X_1, \ldots, X_n$ be an i.i.d. random sample with common density function
$$f(x) = \begin{cases} 3\theta^3 x^{-4} & \text{for } x \geq \theta \\ 0 & \text{otherwise,} \end{cases}$$
where $\theta > 0$ is an unknown parameter.

   (i) Apply the method of moments to obtain an unbiased estimator of θ.

   (ii) Find the maximum likelihood estimator (MLE) of θ and show that it is biased.

   (iii) Which of the above two estimators has a smaller mean squared error (MSE)?

2. (14 points) (5+5+4) Let $X_1, X_2, \ldots, X_n$ from be an i.i.d. random sample Uniform$(0, \theta)$, where where $\theta > 0$ is an unknown parameter. Suppose that we want to test the following hypothesis:
$$H_0 : 3 \leq \theta \leq 4, \qquad \text{versus} \qquad H_1 : \theta < 3 \ \text{ or } \ \theta > 4. \tag{1}$$
Let $Y_n = \max\{X_1, \ldots, X_n\}$. Consider the following two tests:
$$\delta_1 : \ \text{Reject } H_0 \qquad \text{if} \quad Y_n \leq 2.9 \ \text{ or } \ Y_n \geq 4$$
and
$$\delta_2 : \ \text{Reject } H_0 \qquad \text{if} \quad Y_n \leq 2.9 \ \text{ or } \ Y_n \geq 4.5.$$

   (i) Find the power functions of δ_1 and δ_2, when $\theta \leq 4$.

   (ii) Find the power functions of δ_1 and δ_2, when $\theta > 4$.

   (iii) Which of the two tests seems better for testing the hypothesis (1)?

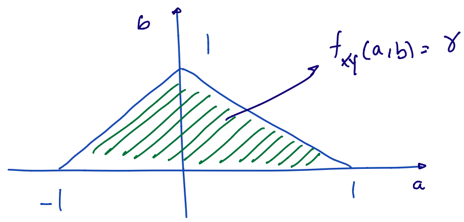
3. (10 points) (5+5) A random sample $X_1, \ldots, X_n$ is drawn from a population with p.d.f.
$$f_\theta(x) = \frac{1}{2}(1 + \theta x), \qquad x \in [-1, 1],$$
and $f_\theta(x) = 0$ if $x \notin [-1, 1]$, where $\theta \in [-1, 1]$ is the unknown parameter.

   (i) Find an unbiased estimator of θ.

   (ii) Is the estimator in (i) consistent? Provide a justification for your answer.

4. (16 points) (4+4+4+4) The joint pdf $f_{XY}(a, b)$ of random variables X and Y is zero outside the triangular region shown below and is equal to a fixed number γ on the triangular region. Answer the following questions:

   (i) Calculate the value of γ.



(ii) Are X and Y independent? Prove your answer and then give an intuitive explanation.

(iii) Calculate $\text{cov}(X, Y)$. Does the result you obtain make sense?

(iv) Calculate the joint CDF of two random variables $Z = X + Y$ and $W = X - Y$.

5. (12 points) (4+4+4) Let X and Y be a pair of random variables with the following distributional specification: $P(Y = 1) = 1 - P(Y = 0) = \alpha$ where $\alpha \in (0, 1)$, and $X|Y = 0 \sim N(0, \sigma^2)$ and $X|Y = 1 \sim N(\mu, \sigma^2)$.

   (i) Find the conditional distribution of Y given X, i.e. $P(Y = 1|X = x)$.

   (ii) Suppose that we have an i.i.d. random sample from this population, i.e. we observe i.i.d. copies (X_i, Y_i), $i = 1, \ldots, n$. Write down the likelihood function and find maximum likelihood estimators $\hat{\alpha}_n$, $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ of α, μ and σ^2.

   (iii) What are the asymptotic distributions of $\hat{\alpha}_n$, $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ (properly standardized)?

6. (12 points) (4+4+4) Suppose $X_1, \cdots, X_n$ are independent, with $X_i \sim N\left(\frac{\theta}{i}, 1\right)$. Here $\theta \in \mathbb{R}$ is an unknown parameter.

   (i) Find an unbiased estimator $\hat{\theta}_n$ for θ which depends on the entire data.

   (ii) Find asymptotic non-degenerate distribution of your estimator, i.e. $d_n(\hat{\theta}_n - \theta)$ converges to a non-degenerate distribution.

   (iii) Suppose that we impose a normal prior $\theta \sim N(0, \tau)$, where $\tau > 0$ is a known constant. Find the posterior distribution of θ given data $X_1, \cdots, X_n$.

7. (12 points) (6+6) Suppose that $A = (a_{ij})_{1 \le i,j \le 2}$ is a 2×2 symmetric matrix, with $a_{11} = a_{22} = \frac{3}{4}$ and $a_{12} = a_{21} = \frac{1}{4}$.

   (i) Find the eigenvalues and eigenvectors of the matrix A.

   (ii) Compute $\lim_{n \to \infty} a_{12}^{(n)}$, where $a_{ij}^{(n)}$ denotes the (i, j)'s entry of matrix A^n.

8. (10 points) Daniel and Ann alternatively toss a fair coin. Daniel tosses the coin first, then Ann tosses the coin, then it is again Daniel's turn and so on. We record the sequence. If there is a head followed by a tail the game ends and the person who tosses the tail wins. What is the probability that Daniel wins the game?

   Hint: Call the event of Daniel winning the game A*, and let* $x = P(A)$*. Also, let* B *denote the event that Daniel sees a Head in the first toss. Use the law of total probability to write down*:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$