"Core Competency" (**Sep. 2019**) **Name (Print):** ____________________
Qualifying Exam (09/03/2019)
Time Limit: 4 hours **Signature:** ____________________

This exam contains 8 problems. Answer **all** of them. Point values are in parentheses. You **must show your work** to get credit for your solutions — correct answers without work will not be awarded points.

No calculators will be allowed in the exam. This is a closed notes/book exam; no cheat sheets are allowed. Normal tables, if needed, will be provided during the exam. Rough paper will be provided to you.

1	8 pts	
2	15 pts	
3	10 pts	
4	18 pts	
5	12 pts	
6	10 pts	
7	10 pts	
8	17 pts	
TOTAL	100 pts	

1. (8 points) (3+5)

   (i) Suppose that X is a nonnegative random variable. Show that

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}X}{t}, \qquad \text{for any } \ t > 0.$$

   (ii) Suppose that $Y \sim N(0,1)$. Show that

$$\mathbb{P}(Y > t) \leq e^{-t^2/2}, \qquad \text{for any } \ t > 0.$$

   Hint: You can assume without proof that $\mathbb{E}[e^{\lambda Y}] = e^{\lambda^2/2}$ for all $\lambda \in \mathbb{R}$.

2. (15 points) (5+5+5)

   (i) Let X be a random variable distributed according to an exponential law with expectation $1/\lambda$, where λ is a positive constant. Given the real positive random variable X, let us define the discrete variable $Y = \lceil X \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, i.e., the function which rounds any real number upwards to the closest integer. For instance, $\lceil 14.3 \rceil = 15$ and $\lceil 14.8 \rceil = 15$. Show that the random variable Y follows a geometric distribution and identify the parameter.

   (ii) Show that for any continuous random variable W with a strictly increasing cumulative distribution function F, we have that $F(W) \sim \text{Uniform}[0, 1]$.

(iii) Using this results of (i) and (ii), propose an algorithm to simulate a realization of the geometric random variable in (i) from $U \sim \text{Uniform}([0,1])$.

3. (10 points) Suppose that X_n and Y_n are independent random variables, where X_n is asymptotically normal with mean 4 and standard deviation $1/\sqrt{n}$, (i.e., $\sqrt{n}(X_n - 4) \xrightarrow{d} N(0,1)$) and Y_n is asymptotically normal with mean 3 and standard deviation $2/\sqrt{n}$. Use the delta method to get an approximate mean and standard deviation of Y_n/X_n.

4. (18 points) (5 + 6 + 7) Let $X_1, \ldots, X_n$ be a random sample (i.i.d.) from a density function f. The corresponding CDF is denoted by F. Denote by $X_{(1)} < \cdots < X_{(n)}$ the order statistics, i.e., a rearrangement of $X_1, \ldots, X_n$ according to their values.

   (i) For $n = 2$, derive the density function of $X_{(1)}$ in terms of f and F. [**Hint**: You may want to find the distribution function of $X_{(1)}$ first.]

   (ii) For $n = 3$, derive the density function of $X_{(2)}$ in terms of f and F.

(iii) For any n and k, derive the density function of $X_{(k)}$ in terms of f and F.

5. (12 points) (6 + 6) Let $X_1, \ldots, X_n$ be the number of accidents at an important intersection in the past n years. We are interested in estimating the probability of zero accidents next year. We model the X_i's as independent random variables distributed according to a Poisson distribution with mean λ.

   (i) Let $q(\lambda)$ be the probability that there will be no accidents next year. Find an unbiased and consistent estimator of $q(\lambda)$.

   (ii) Compute the maximum likelihood estimator of $q(\lambda)$ and derive its asymptotic distribution. Compare this estimator with the one obtained in (i).

6. (10 points) Suppose that $X_1, X_2, \ldots$ are i.i.d. having an exponential distribution with mean 1. Show that

$$\frac{\max_{1 \le k \le n} X_k}{\log n} \xrightarrow{p} 1 \quad \text{as } n \to \infty$$

where $\xrightarrow{p}$ denotes convergence in probability.

7. (10 points) (4 + 6) Suppose that $X_1, \ldots, X_n$ are i.i.d. uniform random variables on $[0, \theta]$ for some $\theta \in [1, 2]$.

   (i) What is the MLE of θ?

   (ii) Suppose that, instead of X_i's, we only observe, for all $i = 1, \ldots, n$,

$$Y_i = \begin{cases} X_i & \text{if } X_i \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   What is the MLE of θ based on $\{Y_1, \ldots, Y_n\}$?

8. (17 points) (8+6+3) Suppose that a measurement Y is recorded with a $N(\theta, \sigma^2)$ sampling distribution, with σ known and θ known to lie in the interval $[0, 1]$ (but otherwise unknown). Consider two point estimators of θ: (a) the posterior mean $\hat{\theta}_B$ based on the assumption of a uniform prior distribution on θ on $[0, 1]$, and (b) the maximum likelihood estimate $\hat{\theta}_M$, restricted to the range $[0, 1]$.

   (i) Show that, as $\sigma \to \infty$, $\hat{\theta}_B$ converges in distribution (to Y_1, say). Identify the limit Y_1. [**Hint**: You may first find the distribution of $\Theta | Y = y$ and then take limits.]

(ii) Show that, as $\sigma \to \infty$, $\hat{\theta}_M$ converges in distribution (to Y_2, say). Identify the limit Y_2.

(iii) If σ is large enough, which estimator $\hat{\theta}_M$ or $\hat{\theta}_B$ has a higher mean squared error, for any value of θ in $[0, 1]$. You may answer this question by comparing the mean squared errors or Y_1 and Y_2 for estimating θ.