"Core Competency" (**May 2021**) **Name (Print):** ____________________
Qualifying Exam (09/13/2021)
Time Limit: 4 hours **Signature:** ____________________

This exam contains 8 problems. Answer **all** of them. Point values are in parentheses. You **must show your work** to get credit for your solutions — correct answers without work will not be awarded points. Please use separate sheet(s) for each question.

No calculators will be allowed in the exam. This is a closed notes/book exam; no cheat sheets are allowed.

1	14 pts	
2	14 pts	
3	10 pts	
4	13 pts	
5	12 pts	
6	12 pts	
7	13 pts	
8	12 pts	
TOTAL	100 pts	

1. (14 points) (5+4+5) Let $X_1, \ldots, X_n$ be an i.i.d. random sample with common density function

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)} & \text{for } x \geq \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(i) Find a one dimensional sufficient statistic T_n.

(ii) Derive the cumulative distribution function F_n of T_n.

(iii) Give an exact $(1-\alpha)$-confidence interval for θ. (Hint: What is the distribution of $F_n(T_n)$?)

2. (14 points) (5+4+5) Let X and Y be two independent exponential random variables with parameters λ and μ, respectively, i.e. $P(X \geq x, Y \geq y) = e^{-\lambda x - \mu y}$, $x \geq 0$, $y \geq 0$. Define random variables

$$T = \min(X, Y) \quad \text{and} \quad \Delta = \begin{cases} 1 & \text{if } X < Y \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the probability density function of T and the probability mass function of Δ.

(ii) Find the joint distribution function of (T, Δ).

(iii) Suppose we have a random sample (T_i, Δ_i), $i = 1, \ldots, n$, i.e. i.i.d. copies of (T, Δ). Write down the likelihood function and find the MLE of λ.

3. (10 points) Let X and Y be two jointly distributed random variables with finite expectations and variances. Show that $Var(Y) = E\{Var(Y|X)\} + Var(E\{Y|X\})$.

4. (13 points) (5+4+4) Suppose $\{U_i\}_{i\geq 1} \overset{i.i.d.}{\sim} U(0,\theta)$, for some $\theta > 0$.

(i) Show that $T_n := \left(\prod_{i=1}^n U_i\right)^{1/n}$ converges in probability to a constant, and find this constant.

(ii) Find a function of T_n that is a consistent estimator for θ.

(iii) Find constants a_n and b_n such that $a_n(T_n - b_n)$ converges to a non-degenerate distribution.

5. (12 points) (4+4+4) Suppose $\{\xi_i\}_{i\geq 0}$ are i.i.d. $N(0,1)$ random variables. Find the constant c such that

$$\frac{\max_{1\leq i\leq n} X_i}{\sqrt{\log n}} \xrightarrow{p} c$$

for each of the following three cases where $\{X_i\}_{i\geq 1}$ is defined.

(i) $X_i = \xi_i$ for $i \geq 1$.

(ii) $X_i = \xi_i + \xi_0$ for $i \geq 1$.

(iii) $X_i = \frac{\xi_i + \xi_{i-1}}{\sqrt{2}}$ for $i \geq 1$.

6. (12 points) Let $A \in \mathbb{R}^{m \times n}$ denote an $m \times n$ matrix with $n < m$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ denote, respectively, the eigenvalues and eigenvectors of $A^T A$. What can we say about ALL the eigenvalues and eigenvectors of AA^T? Justify your answer.

7. (13 points) (3+10) Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F$ (F denotes the CDF). Our goal is to estimate $\gamma = F(0) + 2F(1)$. We employ the following estimate

$$\hat{\gamma} = \frac{1}{n}\left(\sum_{i=1}^{n} \mathbb{I}(X_i \leq 0) + 2\sum_{i=1}^{n} \mathbb{I}(X_i \leq 1)\right),$$

where $\mathbb{I}(\cdot)$ denotes the indicator function.

(i) Calculate $\mathbb{E}(\hat{\gamma})$.

(ii) What is the limiting distribution of $\sqrt{n}(\hat{\gamma} - \gamma)$? Justify your answer.

8. (12 points) (4+4+4) Answer the following questions:

(i) Suppose that $(X_n, Y_n) \overset{d}{\to} N(0, \Sigma)$ in distribution with $\Sigma = [2,1;1,1]$. What does $(X_n - Y_n)^2$ converge to in distribution? Prove your answer.

(ii) Suppose that $(X_n, \sqrt{n}Y_n) \overset{d}{\to} N(0, \Sigma)$ in distribution with $\Sigma = [2,1;1,1]$. What does $(X_n - Y_n)^2$ converge to in distribution? Prove your answer.

(iii) Let $X_n \overset{p}{\to} 1$. For each X_n, we pick Y_n uniformly at random from the interval $[0, X_n]$. What does Y_n converge to in distribution? Prove your answer.