Review Session 1 – Solutions

1 Solutions

1.1 Previous Core Competency Problems

Problem 1 (2018 Summer Practice Problems, # 18)**.** Suppose Σ is a nonnegative definite matrix of $n \times n$ with real entries and real eigenvalues. Show that $\mathrm{Tr}(\Sigma^2) \geq n \cdot \det(\Sigma)^{2/n}$.

Solution

Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of Σ. First, we claim that the eigenvalues of Σ^2 are $\{\lambda_i^2\}_{i=1}^n$. This follows from the fact that $\Sigma v = \lambda v \implies \Sigma^2 v = \lambda \Sigma v = \lambda^2 v$. Furthermore, the algebraic multiplicity of eigenvalue of λ^2 of Σ^2 is the algebraic multiplicity of eigenvalue λ of Σ, as can be seen from the factorization

$$\det(\Sigma^2 - \lambda^2 \,\mathrm{Id}) = \det(\Sigma - \lambda \,\mathrm{Id}) \det(\Sigma + \lambda \,\mathrm{Id}).$$

Thus, the eigenvalues of Σ^2 is precisely the set $\{\lambda_i^2\}_{i=1}^n$. Then, since $\mathrm{Tr}(\cdot)$ (resp., $\det(\cdot)$) sums (resp., multiplies) the eigenvalues, weighted by their algebraic multiplicies, we have

$$\mathrm{Tr}(\Sigma^2) \geq n \det(\Sigma)^{2/n} \iff \sum_{i=1}^n \lambda_i^2 \geq n \sqrt[n]{\prod_{i=1}^n \lambda_i^2} \iff \frac{1}{n}\sum_{i=1}^n \lambda_i^2 \geq \sqrt[n]{\prod_{i=1}^n \lambda_i^2}.$$

However, this last inequality is just the AM-GM inequality.

> **Remark.** We must assume Σ has *real* eigenvalues. As a counterexample, $\Sigma = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$ is verifiably positive-definite, but has complex eigenvalues. Then, $\Sigma^2 = \begin{pmatrix} 1 & 3 \\ -9 & -2 \end{pmatrix}$. Yet, $\mathrm{Tr}(\Sigma^2) = -1$ and $\det(\Sigma) = 5$ meaning the desired inequality is not true. However, a more general bound on the determinant still holds, called Hadamard's inequality.

Problem 2 (2020 September Exam, # 8)**.** For every $n \geq 1$, let A_n be an $n \times n$ symmetric matrix with non negative entries. Let $R_n(i) := \sum_{j=1}^n A_n(i,j)$ denote the ith row/column sum of A_n. Assume that

$$\lim_{n\to\infty} \max_{1\leq i \leq n} |R_n(i) - 1| = 0.$$

Let $\lambda_n \geq 0$ denote an eigenvalue with the largest absolute value, and let $\mathbf{x} := (x_1, \ldots, x_n)$ denote its corresponding eigenvector.

(a) Show that $\frac{1}{n}\sum_{i,j=1}^n A_n(i,j) \to 1$.

(b) Show that $\lambda_n |x_i| \leq \max_{1\leq j\leq n} |x_j| R_n(i)$.

(c) Using parts (a) and (b), show that $\lambda_n \to 1$.

Solution

(a)

$$\left| \frac{1}{n}\sum_{i,j=1}^n A_n(i,j) - 1 \right| = \left| \frac{1}{n}\sum_{i=1}^n R_n(i) - 1 \right| \leq \max_{1\leq i\leq n} |R_n(i) - 1| \to 0.$$

(b) $\lambda \mathbf{x} = A_n \mathbf{x}$ gives us $\lambda_n x_i = \sum_{j=1}^n A_n(i,j) x_j$ for all $i \in [n]$. Then,

$$\lambda_n |x_i| \le \sum_{j=1}^n A_n(i,j)|x_j| \le \max_{1\le j\le n} |x_j| \cdot R_n(i).$$

(c) Using the fact that $\lambda_n = \sup_{\mathbf{y}:\|\mathbf{y}\|_2=1} \mathbf{y}^T A_n \mathbf{y}$, we have from part (a):

$$\lambda_n = \sup_{\mathbf{y}:\|\mathbf{y}\|_2=1} \sum_{i,j=1}^n A_n(i,j) y_i y_j \ge \frac{1}{n}\sum_{i,j} A_n(i,j) \to 1.$$

On the other hand, by part (b):

$$\lambda_n \le \max_{1\le j\le n} R_n(j) \to 1.$$

Thus, combining these two estimates, $\lambda_n \to 1$.

Problem 3 (2021 May Exam, # 7)**.** Suppose that $A = (a_{ij})_{1\le i,j\le 2}$ is a 2×2 symmetric matrix, with $a_{11} = a_{22} = \frac{3}{4}$ and $a_{12} = a_{21} = \frac{1}{4}$.

1. Find the eigenvalues and eigenvectors of the matrix A.
2. Compute $\lim_{n\to+\infty} a_{12}^{(n)}$, where $a_{ij}^{(n)}$ denotes the (i,j)'s entry of matrix A^n.

Solution

1. We have
$$\det(\lambda\,\mathrm{Id} - A) = 0 \iff (\lambda - 3/4)^2 - 1/16 = 0 \iff 2\lambda^2 - 3\lambda + 1 \iff \lambda = 1, 1/2.$$
2. The eigenspace of eigenvalue $\lambda = 1$ is spanned by $(1,1)$ and the eigenspace of eigenvalue $\lambda = 1/2$ is spanned by $(1,-1)$. Thus, A has eigendecomposition
$$A = Q^T D Q := \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$
Thus,
$$A^n = Q^T D^n Q = Q^T \begin{pmatrix} 1 & 0 \\ 0 & 1/2^n \end{pmatrix} Q.$$
Using this, we see $a_{12}^{(n)} = 1/2 - 1/2^{n+1} \overset{n\to\infty}{\longrightarrow} 1/2$.

Problem 4 (2021 Sept Exam, # 6)**.** Let $A \in \mathbb{R}^{m\times n}$ denote an $m \times n$ matrix with $n < m$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ denote, respectively, the eigenvalues and eigenvectors of $A^T A$. What can we say about ALL the eigenvalues and eigenvectors of AA^T? Justify your answer.

Solution

From looking at the SVD's of A and A^T, we conclude $A^T A$ and AA^T have the same eigenvalues. Furthermore, by the SVD, the eigenvectors $\{\mathbf{u}_i\}_i$ of AA^T are related to the eigenvectors $\{\mathbf{v}_i\}_i$ of $A^T A$ via the SVD relation $A\mathbf{v}_i = \mathbf{u}_i\sqrt{\lambda_i}$.

1.2 Additional Practice

Problem 5. Let A be a 3×3 real-valued matrix such that $A^T A = AA^T = \mathrm{Id}_3$ and $\det(A) = 1$. Prove that 1 is an eigenvalue of A.

Solution

Let λ be an eigenvalue of A with eigenvector v. Write $\|v\|_2^2 = \|A^T A v\|^2 = v^T(A^T A)^T A^T A v = \lambda v^T(AA^T)\lambda v = \lambda^2\|v\|_2^2$ meaning $\lambda^2 = 1 \implies \lambda = \pm 1$. Since the characteristic polynomial is degree 3 and the product of the eigenvalues is 1, this implies 1 must be an eigenvalue of A.

Problem 6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric $n \times n$ matrix such that $\mathrm{Tr}(A^2) = 0$. Show that $T = \mathbf{0}_{n\times n}$. Hint: use the fact that $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ for matrices A, B, C.

Solution

Write the eigendecomposition $A = QDQ^T$ and observe

$$0 = \mathrm{Tr}(A^2) = \mathrm{Tr}(QD^2Q^T) = Tr(Q^TQD^2) = \mathrm{Tr}(D^2) = \lambda_1^2 + \cdots + \lambda_n^2.$$

This implies all $\lambda_i = 0$ and thus $D \equiv \mathbf{0}_{\times n}$ which means T is also zero.

Problem 7. Let matrices $A, B \in \mathbb{R}^{n\times n}$ have respective eigendecompositions $Q_1D_1Q_1^T$ and $Q_2D_2Q_2^T$ (recall this means each D_i is a diagonal matrix of eigenvalues and each Q_i is an orthogonal matrix). Prove that $Q_1 = Q_2$ if and only if $AB = BA$. You may assume that A, B do not have any repeated eigenvalues.

Solution

In the forward direction, if $Q = Q_1 = Q_2$, then

$$AB = QD_1Q^TQD_2Q^T = QD_1D_2Q^T,$$

and similarly $BA = QD_2D_1Q^T$. But, diagonal matrices always commute so that $D_1D_2 = D_2D_1$. Thus, $AB = BA$.
In the other direction, if $AB = BA$, then let v, λ be an eigenvector, eigenvalue pair of A, i.e. $Av = \lambda v$. Then,

$$ABv = BAv = B\lambda v = \lambda Bv.$$

This implies both v and Bv are eigenvectors of A. Since the eigenspace of λ is one-dimensional (because A has no repeated eigenvectors), this means $v \propto Bv$ so that v is an eigenvector of B. Then, we conclude A and B have the same eigenvectors which means $Q_1 = Q_2$.

Problem 8. Let $A = uv^T \in \mathbb{R}^{n\times n}$ be a *rank-one* matrix, i.e. $u, v \in \mathbb{R}^n$. Suppose $u, v \neq \mathbf{0}_n$. Find, with proof, all the eigenvalues of A.

Solution

We claim 0 and v^Tu are the only eigenvalues of A. 0 is an eigenvalue since A is not full rank. v^Tu is an eigenvalue since

$$Au = u(v^Tu) = (v^Tu)\cdot u.$$

Furthermore, we claim the geometric multiplicity of the eigenvalue 0 is $n-1$ and hence there can be no other eigenvalues. This is true since any vector v' in the orthogonal complement of v satisfies $Av' = uv^Tv' = u(v^Tv') = \mathbf{0}_n$. Since this orthogonal complement $\mathrm{Span}(v)^\perp$ has dimension $n-1$ the eigenspace of eigenvalue 0 has dimension $n-1$ and so our claim is proven.

Problem 9 (Heisenberg uncertainty principle). Suppose $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices satisfying $AB + BA = \mathrm{Id}_n$. Show that for all vectors $v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$,

$$\max\left\{\frac{\|Av\|_2}{\|v\|_2}, \frac{\|Bv\|_2}{\|v\|_2}\right\} \geq 1/\sqrt{2}.$$

Solution

Write $v^T v = v^T \operatorname{Id}_n v = v^T ABv + v^T BAv \le 2|(v^T A) \cdot (Bv)| \le 2\|Av\|_2 \|Bv\|_2$ by Cauchy-Schwarz. Thus,

$$\|v\|_2^2 \le 2\|Av\|_2 \|Bv\|_2,$$

meaning one of $\|Av\|_2/\|v\|_2$ or $\|Bv\|_2/\|v\|_2$ is larger than $1/\sqrt{2}$.

Problem 10. Let $A = (a_{i,j})$ be a $n \times n$ real matrix whose diagonal entries $a_{i,i}$ satisfy $a_{i,i} \ge 1$ for all $i \in 1, \dots, n$. Suppose also that

$$\sum_{i \ne j} a_{i,j}^2 < 1.$$

Prove that the inverse matrix A^{-1} exists.

Solution

For contradiction, suppose A^{-1} does not exist which mean A has a non-trivial kernel, i.e. $\exists v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$ such that $Av = 0$. WLOG, let $\|v\|_2 = 1$. Then, $Av = 0$ implies

$$\forall i \in [n] : a_{ii} \cdot v_i + \sum_{j : j \ne i} a_{ij} \cdot v_j = 0 \implies a_{ii} \cdot v_i = -\sum_{j : j \ne i} a_{ij} \cdot v_j.$$

Thus, squaring both sides and summing over $i \in [n]$, we obtain

$$\sum_{i=1}^n a_{ii}^2 v_i^2 = \sum_{i=1}^n \left(\sum_{j : j \ne i} a_{ij} \cdot v_j \right)^2.$$

By Cauchy-Schwarz the RHS is at most

$$\sum_{i=1}^n \left(\sum_{j : j \ne i} a_{ij}^2 \right) \|v\|_2^2 = \sum_{i,j : i \ne j} a_{ij}^2 < 1.$$

On the other hand, $\sum_{i=1}^n a_{ii}^2 v_i^2 \ge \sum_{i=1}^n v_i^2 = \|v\|_2^2 = 1$, which is a contradiction to the above.

Problem 11 (Greshgorin circle theorem)**.** Let $A \in \mathbb{C}^{n \times n}$ with entries a_{ij}. For $i \in \{1, \dots, n\}$ let R_i be the sum of the absolute values of the non-diagonal entries in the i-th row:

$$R_i := \sum_{j \ne i} |a_{ij}|.$$

Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i, called a *Gershgorin disc*. Show that every eigenvalue of A lies within at least one of the Gershgorin discs.

Solution

Let λ, v be an eigenvalue/eigenvector pair of A. Suppose v_i has the largest modulus among $\{v_1, \dots, v_n\}$, the entries of v, or $|v_i| = \max_j |v_j| \ne 0$. Then, let $u = v/v_i$. Then, each u_j has modulus $|u_j| = \frac{|v_j|}{|v_i|} \le 1$ while $u_i = 1$. Now, u of course is a valid eigenvector of v: $Au = \lambda u$. In particular, looking at the i-th row, we have

$$\sum_j a_{ij} u_j = \lambda u_i = \lambda.$$

Splitting this sum gives us

$$\sum_{j \ne i} a_{ij} u_j + a_{ii} = \lambda \implies |\lambda - a_{ii}| = \left| \sum_{j \ne i} a_{ij} u_j \right| \le \sum_{j \ne i} |a_{ij}| |u_j| \le \sum_{j \ne i} |a_{ij}| = R_i,$$

where we use the fact that $|u_j| \le 1$ for $j \ne i$.

Problem 12. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$. Find $\lim_{n\to\infty} A^n$. Hint: the eigenvalues of a lower triangular matrix are its diagonal entries.

Solution

The eigenvalues of A are $1, 1/2, 1/3$ as can be seen from the fact that $\det(A - \lambda I) = (\lambda - 1)(\lambda - 1/2)(\lambda - 1/3)$. Then, via direct computation, the eigenspace of eigenvalue 1 is spanned by $(1,1,1)$, that of eigenvalue $1/2$ spanned by $(0,1,2)$, and that of eigenvalue $1/3$ spanned by $(0,0,1)$. Then, letting

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix},$$

we have that $AP = PD$ where $D = \text{diag}(1, 1/2, 1/3)$. In fact, P is invertible since it has non-zero determinant so that $A = PDP^{-1}$. Then,

$$A^n = PD^nP^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2^n & 0 \\ 0 & 0 & 1/3^n \end{pmatrix} P^{-1}.$$

As $n \to \infty$, the middle matrix above goes to a matrix with just $(1,1)$-entry 1 and all other entries 0. Let p_{11}^{-1} be the $(1,1)$-th entry of P^{-1}. Then the above RHS in the limit is

$$P \begin{pmatrix} p_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} p_{11}^{-1} & 0 & 0 \\ p_{11}^{-1} & 0 & 0 \\ p_{11}^{-1} & 0 & 0 \end{pmatrix}.$$

However, we can compute p_{11}^{-1} without computing the entire inverse P^{-1} since we know $PP^{-1} = \text{Id}_{3\times 3}$ means that $p_{11}^{-1} = 1$. Thus

$$\lim_n A_n = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 13. Let A, B be $n \times n$ matrices. Show that BA and AB have the same eigenvalues if A is invertible

Solution

The characteristic polynomial of AB is

$$\det(AB - \lambda\,\text{Id}) = \det(A^{-1}A(AB - \lambda\,\text{Id})) = \det(A^{-1}(AB - \lambda\,\text{Id})A) = \det(BA - \lambda\,\text{Id}).$$

Thus, AB and BA have the same characteristic polynomial meaning they have the same eigenvalues.

Problem 14. Let $A = (a_{ij})$ be a 2×2 real matrix such that

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 < \frac{1}{1000}.$$

Prove that $\text{Id}_{2\times 2} + A$ is invertible.

Solution

For contradiction, suppose otherwise meaning $\text{Id} + A$ has a non-trivial kernel, i.e. $(\text{Id} + A)v = 0$ for some $v \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, this means

$$v_1 + a_{11}v_1 + a_{21}v_2 = 0$$
$$v_2 + a_{12}v_1 + a_{22}v_2 = 0.$$

Thus,

$$\|v\|_2^2 = (a_{11}v_1 + a_{21}v_2)^2 + (a_{12}v_1 + a_{22}v_2)^2 \le \|v\|_2^2 \left(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2\right) \le \frac{1}{1000}\|v\|_2^2.$$

This is only possible if $\|v\|_2 = 0$ meaning $v = (0, 0)$, a contradiction.

Problem 15. Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric $n \times n$ matrix and let $\lambda_1 \ge \cdots \ge \lambda_n$ be its eigenvalues in decreasing order. Show that

$$\lambda_k \le \max_{U:\dim(U)=k} \min_{x\in U:\|x\|_2=1} \langle Ax, x\rangle.$$

The maximum above is over all k-dimensional subspaces U of $\mathbb{R}^n$. Hint: form an orthonormal basis of eigenvectors to make U.

Solution

Choose orthonormal eigenvectors $v_1, \dots, v_k$ corresponding to $\lambda_1, \dots, \lambda_k$ (they can be orthonormalized by Gram-Schmidt). Then, let $U = \mathrm{Span}(v_1) \oplus \cdots \oplus \mathrm{Span}(v_k)$. Then, for any $x \in U$, we can represent $x = \sum_{k=1}^{k} \alpha_i v_i$ for coordinates $\alpha_i \in \mathbb{R}$. Suppose $\|x\|_2 = 1$. Then,

$$\langle Ax, x\rangle = \left\langle \sum_{i=1}^{k} \alpha_i v_i \lambda_i, \sum_{i=1}^{k} \alpha_i v_i \right\rangle = \sum_{i=1}^{k} \alpha_i^2 \lambda_i \ge \sum_{i=1}^{k} \alpha_i^2 \lambda_k = \lambda_k,$$

where the last equality follows from $\sum_{i=1}^{k} \alpha_i^2 = \|x\|_2 = 1$.

Problem 16. For a vector $v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$, define the map $F : \mathbb{R}^n \to \mathbb{R}^n$ via $F(x) = \operatorname*{argmin}_{z\in\mathrm{Span}(v)} \|z - x\|_2$. Compute F explicitly in terms of v. Is $F : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation?

Solution

In the minimization, we can instead optimize $\|cv - x\|_2^2$ over $c \in \mathbb{R}$. Expanding the square we have

$$\|cv - x\|_2^2 = c^2 v^T v - 2cv^T x + x^T x.$$

This is a convex quadratic in c. Thus, taking derivative w.r.t. c and setting equal to 0, we find the minimizer is $c = \frac{v^T x}{v^T v}$. Thus, $F(x) = \frac{vv^T}{v^T v} \cdot x$. This is a linear transformation since it's just a matrix times x.

Problem 17. Suppose $A \in \mathbb{R}^{n\times n}$ and $A = A^T$ with all eigenvalues of A being positive. Show there exists a matrix B such that $B^2 = A$.

Solution

By the eigendecomposition, we have $A = QDQ^T$ for D a diagonal matrix of A's positive eigenvalues $\lambda_1, \dots, \lambda_n > 0$. Let $C = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ and let $B = QCQ^T$. Then, $B^2 = QC^2Q^T = QDQ^T = A$.

Problem 18. Let $A \in \mathbb{R}^{n\times n}$ and let $\{\sigma_i\}_{i=1}^n$ be the singular values of A. Show that $|\det(A)| = \prod_{i=1}^n \sigma_i$.

Solution

This follows from the SVD $A = U\Sigma V^T$ and the fact that U, V are orthogonal so that $\det(A) = \det(U)\det(\Sigma)\det(V^T) = \pm\det(\Sigma)$.

Problem 19. (low-rank matrix approximation) Let $A \in \mathbb{R}^{m\times n}$ matrix and for a positive integer $p < \mathrm{rank}(A)$, define $A_p = \sum_{i=1}^p \sigma_i u_i v_i^T$ where σ_i is the i-th (largest) singular value of A, and u_i, v_i are respective left/right singular vectors, i.e. the SVD is $A = U\Sigma V^T$. Then, prove that

$$\sup_{x:\|x\|_2=1} \|(A - A_p)x\|_2 = \sigma_{p+1}.$$

Solution

Observe that since $A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$, we have $U^T(A - A_p)V = \text{diag}(0, \ldots, 0, \sigma_{p+1}, \ldots)$. By the orthogonal invariance of the 2-norm, we have

$$\|(A - A_p)x\|_2 = \|U^T(A - A_p)Vx\|_2.$$

which is the top singular value of $U^T(A - A_p)V$ or σ_{p+1}.

Problem 20. Suppose $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix that satisfies $P^2 = P$, a so-called *idempotent* matrix. Find all the eigenvalues of P with their (algebraic) multiplicities in terms of P.

Solution

Let v be an eigenvector of P that corresponds to eigenvalue λ. We have

$$P^2 = P \implies P^2 v = Pv = \lambda v.$$

However, $P^2 v = P(Pv) = P(\lambda v) = \lambda P v = \lambda^2 v$. Thus, $\lambda^2 v = \lambda v$ for every eigenvector. Since $v \neq 0$, this means $\lambda^2 = \lambda \implies \lambda \in \{0, 1\}$. Now, P admits an eigendecomposition $P = Q\Lambda Q^T$. We also have since multiplication by an invertible matrix doesn't change rank, $\text{rank}(P) = \text{rank}(Q\Lambda Q^T) = \text{rank}(\Lambda)$. However, $\text{rank}(\Lambda)$, and hence $\text{rank}(P)$, is exactly the multiplicity of eigenvalue $\lambda = 1$, while eigenvalue $\lambda = 0$ then has multiplicity $n - \text{rank}(P)$.

Problem 21. Suppose that Σ is the covariance matrix of k zero-mean random variables $X_1, \ldots, X_k$, i.e. if $X = (X_1, \ldots, X_k)$ then $\Sigma := \mathbb{E}[XX^T]$. Prove that if Σ is singular, then $X_1, \ldots, X_k$ are linearly dependent almost everywhere.

Solution

We know that Σ is singular, so it has at least one zero eigenvalue since $\det(\Sigma) = 0$ is the product of the eigenvalues. Let's call its corresponding eigenvector q. Since $\Sigma q = 0$, we have $q^T \Sigma q = 0$. then,

$$q^T \Sigma q = 0 \implies q^T \mathbb{E}[XX^T]q = \mathbb{E}[q^T XX^T q] = \mathbb{E}[(X^T q)^2] = 0.$$

Since $(X^T q)^2$ is a nonnegative random variable with mean zero, it must be zero almost everywhere.