# Review Session 2 – Solutions

## 1 Solutions

### 1.1 Previous Core Competency Problems

**Problem 1** (2018 Summer Practice, # 4)**.** Consider a routine screening test for a disease. Suppose the frequency of the disease in the population (base rate) is $0.5\%$. The test is highly accurate with a $5\%$ false positive rate and a $10\%$ false negative rate [a false positive happens when a test result indicates that the disease is present (the result is positive), but it is, in fact, not present. Similarly, a false negative happens when a test result indicates that the disease is not present (the result is negative), but it is, in fact, present].

You take the test and it comes back positive. What is the probability that you have the disease?

**Solution**

We proceed by Bayes' Theorem. Let $P$ denote test being positive. Let $D$ denote disease being present and $D^c$ denote disease being absent. We have

$$\mathbb{P}(D|P) = \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|D^c)\mathbb{P}(D^c)} = \frac{(1-\mathbb{P}(P^c|D))\mathbb{P}(D)}{(1-\mathbb{P}(P^c|D))\mathbb{P}(D) + \mathbb{P}(P|D^c)\mathbb{P}(D^c)} = \frac{0.9\cdot 0.005}{0.9\cdot 0.005 + 0.05\cdot 0.995}$$

**Problem 2** (2018 Summer Practice, # 4)**.** Suppose that the radius of a circle is a random variable having the following probability density function:

$$f(x) = \frac{1}{8}(3x+1), \qquad 0 < x < 2$$

and $0$ otherwise. Determine the probability density function of the area of the circle.

**Solution**

The function $g : (0,2) \to (0,4\pi)$ defined by $g(x) = \pi \cdot x^2$ is a monotone bijection so that if $Y = \pi X^2$, where $f_X(x) = \frac{1}{8}(3x+1)\mathbf{1}_{(0,2)}(x)$, then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{8}(3\sqrt{y/\pi}+1)\cdot \mathbf{1}_{(0,4\pi)}(y)\cdot \frac{y^{-1/2}}{2\sqrt{\pi}}$$

**Problem 3** (2018 Summer Practice, # 9)**.** Suppose $X_1, X_2$ are i.i.d. random variables from a distribution $F$ with mean $0$ and variance $1$.

(a) If $F = N(0,1)$, show that $\frac{X_1+X_2}{\sqrt{2}} \overset{\text{d}}{=} X_1$.

(b) Now suppose that $\frac{X_1+X_2}{\sqrt{2}} \overset{\text{d}}{=} X_1$. Show that $F = N(0,1)$.

**Solution**

Let $M_Z(t)$ denote the mgf of an R.V. $Z$. Then, we have since $X_1 + X_2 \sim \mathcal{N}(0,2)$,

$$M_{(X_1+X_2)/\sqrt{2}}(t) = \exp\left(\frac{t^2}{2}\right)$$

which is the mgf of a standard normal. Now, we show the other direction. It suffices to show the functional equation $M_{X_1}(t) = M_{X_1}(t/\sqrt{2})^2$ has unique solution $e^{t^2/2}$.

Let $f(t) = \log M_{X_1}(t)$ so that our functional equation becomes $f(t) = 2f(t/\sqrt{2})$ meaning $f''(t) = f''(t/\sqrt{2})$. Thus, we observe $f''$ is constant since $f''(x) = f''(0)$ for any $x$. This means $f$ is a second degree polynomial or $f(t) = a + bt + ct^2$. Next, we solve for $a, b, c \in \mathbb{R}$. First, we have $a = 0$ since any mgf $M_Z(t)$ satisfies $M_z(0) = 1 \implies \log M_z(0) = 0$. Thus, $M_{X_1}(t) = e^{bt+ct^2}$. Next,

$$M'_{X_1}(0) = \mathbb{E}[X_1] = 0 = (b + 2ct)e^{bt+ct^2}|_{t=0} = b.$$

Similarly,

$$M''_{X_1}(0) = \mathbb{E}[X_1^2] = 1 = 2c \cdot e^{ct^2} + (2ct)^2 e^{ct^2}|_{t=0} = 2c \implies c = 1/2.$$

Thus, $M_{X_1}(t) = e^{t^2/2} \implies X_1 \sim \mathcal{N}(0, 1)$.

Alternatively, here is a quicker, but tricky, proof of this direction. Let

$$S_{2^n} := \frac{1}{2^{n/2}} (X_1 + \cdots + X_{2^n})$$

where $\{X_i\}_{i=1,\ldots,n}$ are i.i.d. $X_1$. Then, using $\frac{X_1+X_2}{\sqrt{2}} \stackrel{\text{d}}{=} X_1$, we have $S_{2^n} \stackrel{\text{d}}{=} S_{2^{n-1}}$ for all $n \in \mathbb{N}$. But $S_{2^n} \xrightarrow{\text{d}} \mathcal{N}(0, 1)$ as $n \to \infty$ by CLT meaning $X_1 \sim N(0, 1)$.

**Problem 4** (2018 Summer Practice, # 14). Suppose $f : [0, \infty) \to \mathbb{R}$ is a function such that $f(x + y) = f(x)f(y)$.

(a) Show that $f(x) \geq 0$ for all real $x \geq 0$.

(b) Show that $f(0) \in \{0, 1\}$.

(c) Show that for any nonnegative rational number $r$ one has $f(r) = c^r$, where $c \in [0, \infty)$.

(d) If $f$ is assumed to be continuous, show that $f(x) = c^x$ for all real $x \geq 0$.

(e) Suppose $X$ is a nonnegative random variable such that

$$\mathbb{P}(X > s + t) = \mathbb{P}(X > s)\mathbb{P}(X > t)$$

for every $s, t \geq 0$. If $X$ has a continuous distribution function, name the distribution of $X$.

**Solution**

(a) $f(x) = f(x/2)^2 \geq 0$ for $x \geq 0$.

(b) $f(0) = f(0)^2 \implies f(0) \in \{0, 1\}$.

(c) Using the equation, we have $f(p/q) = f(1/q)^p = f(1)^{p/q}$ for $p, q \in \mathbb{N}$.

(d) This immediate from (c) since $\mathbb{Q}$ is dense in $\mathbb{R}_{\geq 0}$.

(e) Letting $f(x) := 1 - F_X(x)$ where $F_X(\cdot)$ is the cdf of $X$, we have from (a)–(d) that $X$ must be an exponential distribution.

**Problem 5** (2018 Summer Practice, # 15). Suppose $X$ is a random variable taking values in $[0, 1]$.

(a) Show that $\mathrm{Var}(X) \leq 1/4$.

(b) Find a random variable $X$ for which equality holds in part (a).

**Solution**

Recall that $g(t) = \mathbb{E}[(X - t)^2]$ has a minimum at $t = \mathbb{E}[X]$. Thus,

$$g(1/2) \geq \mathrm{Var}(X)$$

However, $g(1/2) = 1/4 + \mathbb{E}[X^2] - \mathbb{E}[X]$. However, $X^2 \leq X \implies \mathbb{E}[X^2] \leq \mathbb{E}[X] \implies g(1/2) \leq 1/4$. Thus, $\mathrm{Var}(X) \leq 1/4$ for $X$

with support $[0, 1]$.

The bound is sharp for $X \sim$ Bernoulli$(1/2)$.

**Remark 1.1.** This is a specific case of Popoviciu's inequality on variances.

**Problem 6** (2018 September, # 2)**.** We consider balls of random radius $R$.

(i) Suppose that $R$ is uniformly distributed on $[1, 10]$. Find the probability density function of the volume $V$ of a ball (Recall that $V = \frac{4}{3}\pi R^3$).

(ii) Suppose that $R$ has a log-normal distribution, meaning that $\log(R) \sim \mathcal{N}(\mu, \sigma^2)$ for some parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. Show that $V$ also has a log-normal distribution and find its parameters.

**Solution**

(i) Let $g(r) = \frac{4}{3}\pi r^3$. Then, $g^{-1}(v) = \left(\frac{3}{4\pi} \cdot v\right)^{1/3}$ which has derivative $\frac{\partial}{\partial v} g^{-1}(v) = \left(\frac{3}{4\pi}\right)^{1/3} \left(\frac{1}{3}\right) \frac{1}{v^{2/3}}$. Thus,

$$f_V(v) = f_R(r) \cdot \left|\frac{\partial}{\partial v} g^{-1}(v)\right| = \frac{1}{9} \cdot \mathbf{1}\left\{\left(\frac{3}{4\pi} \cdot v\right)^{1/3} \in [1, 10]\right\} \cdot \frac{1}{(4\pi)^{1/3} 3^{2/3} v^{2/3}}.$$

(ii) We have $\log(V) = \log\left(\frac{4}{3}\pi\right) + 3\log(R) \sim \mathcal{N}(3\mu + \log(4\pi/3), \sigma^2)$.

**Problem 7** (2018 September, # 4)**.** (i) Let $X$ be a random variable and $a \in \mathbb{R}$. Show that (using Markov's inequality or otherwise):

$$\mathbb{P}[X \geq a] \leq \inf_{s \geq 0} e^{-sa} \mathbb{E}[e^{sX}].$$

(ii) Let $N$ be a Poisson random variable with parameter $\lambda > 0$; i.e.,

$$\mathbb{P}[N = n] = e^{-\lambda} \frac{\lambda^n}{n}, \qquad n \geq 0.$$

Show that $\mathbb{E}[e^{sN}] = e^{\lambda(e^s - 1)}$ for all $s \in \mathbb{R}$.

(iii) Let $N$ be as in (ii) and let $m \geq \lambda$ be an integer. Use (i) and (ii) to show that

$$\mathbb{P}[N \geq m] \leq \left(\frac{\lambda}{m}\right)^m e^{m - \lambda}.$$

**Solution**

(i) We have for any $s \geq 0$, by Markov's inequality:

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{sX} \geq e^{sa}) \leq e^{-sa} \mathbb{E}[e^{sX}].$$

Taking infimum over $s$ we are done.

(ii) This is just the mgf of a Poisson R.V.. We have

$$\mathbb{E}[e^{sN}] = \sum_{n=0}^{\infty} \frac{\lambda^n \cdot e^{-\lambda}}{n!} \cdot e^{sn} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda \cdot e^s)^n}{n!} = e^{\lambda e^s - \lambda}.$$

(iii) Plugging in (ii) into (i), we have

$$\mathbb{P}(N \geq m) \leq \inf_{s \geq 0} \exp(\lambda(e^s - 1) - sm).$$

It's straightforward to verify the function $s \mapsto \lambda e^s - sm$ is minimized at $s = \log(m/\lambda) \geq 0$. At this particular value of $s$, the bound above becomes

$$e^{-\lambda} \exp(-m \log(m/\lambda) + m) = \left(\frac{\lambda}{m}\right)^m e^{m-\lambda}.$$

**Problem 8** (2019 May, # 4)**.** We model the lifetime of a device as a random variable $T \geq 0$ with c.d.f. $F(t)$ and density $f(t)$. Suppose that $f(t)$ is continuous for $t \geq 0$ and define the intensity of failure as

$$\lambda(t) = \lim_{h \downarrow 0} \frac{P[t \leq T \leq t + h | T \geq t]}{h} \qquad \text{for } t \geq 0.$$

(a) Express $\lambda(t)$ through $f(t)$ and $F(t)$.

(b) Compute the intensity of failure when $T \sim$ Exp$(\alpha)$, $\alpha > 0$.

(c) Show that $F(t) = 1 - \exp\{-\int_0^t \lambda(s)\, ds\}$ for $t \geq 0$.

(d) Determine $F(t)$ and $f(t)$ in the case that $\lambda(t) = \alpha t^\gamma$ for some $\alpha > 0$ and $\gamma > 0$.

**Solution**

(a) We have

$$\frac{\mathbb{P}(t \leq T \leq t + h | T \geq t)}{h} = \frac{\mathbb{P}(t \leq T \leq t + h)}{\mathbb{P}(T \geq t) \cdot h} = \frac{F(t+h) - F(t)}{h} \cdot \frac{1}{1 - F(t)}.$$

Thus, taking the limit $h \downarrow 0$, we get $\lambda(t) = \frac{f(t)}{1-F(t)}$.

(b) In this case, we have $f(t) = \alpha \exp(-\alpha t) \cdot \mathbf{1}\{t \geq 0\}$ and $F(t) = (1 - \exp(-\alpha t)) \cdot \mathbf{1}\{t \geq 0\}$. Thus, plugging into the formula from (a), we have $\lambda(t) = \alpha \cdot \mathbf{1}\{t \geq 0\}$.

(c) Rearranging the desired result and taking $\log$, it suffices to show

$$-\int_0^t \lambda(s)\, ds = -\int_0^t \frac{f(s)}{1 - F(s)}\, ds = \log(1 - F(t)),$$

which is immediate from inspecting the derivative of the RHS with respect to $t$.

(d) $F(t) = 1 - \exp\left(-\frac{\alpha}{\gamma+1} \cdot t^{\gamma+1}\right)$ can be found using the formula from (c) with $\lambda(t) = \alpha \cdot t^\gamma$ plugged in. Then, $f(t) = F'(t) = \alpha t^\gamma \exp\left(-\frac{\alpha}{\gamma+1} \cdot t^{\gamma+1}\right)$.

**Problem 9** (2019 May, # 5)**.** You are working as a TA in the help room for a duration $t$. The number of students arriving during that period is Poisson distributed with parameter $t\lambda$. For each student, the time $T$ to answer their questions is exponentially distributed with parameter $\alpha$ and this time is independent of all other students. Prove that the distribution of the number $X$ of students that arrive while you are busy with one fixed (randomly chosen) student is geometric with some parameter $p$ and determine $p$ in terms of $\alpha$ and $\lambda$.

*Hint:* The formula $\int_0^\infty s^k e^{-s}\, ds = k!$ for $k = 0, 1, 2, \ldots$ can be used without proof.

**Solution**

$$\mathbb{P}(X = k) = \int_0^\infty \mathbb{P}(X = k | T = t) \cdot \left(\alpha e^{-\alpha t}\right)\, dt = \int_0^\infty \frac{(t\lambda)^k}{k!} e^{-t\lambda} \cdot \left(\alpha e^{-\alpha t}\right)\, dt = \frac{\alpha \lambda^k}{k!(\alpha + \lambda)^{k+1}} \int_0^\infty s^k e^{-s}\, ds.$$

Substituting the hint's formula, the above RHS becomes $\frac{\alpha}{\alpha+\lambda} \cdot \left(\frac{\lambda}{\alpha+\lambda}\right)^k$. Thus, $X$ is geometric with parameter $p = \lambda/(\alpha + \lambda)$.

**Problem 10** (2019 May, # 7)**.** Suppose you have $n$ red balls and one blue ball. We will do two experiments.

(a) In the first experiment, you first drop the $n$ red balls uniformly on the interval $[0,1]$, independent of each other. Having done this, now you drop the blue ball uniformly in the interval $[0,1]$, independent of previous ball drops. Let $X$ denote the number of red balls to the left of the blue ball. Find $\mathbb{P}(X=k)$, for $k=0,\ldots,n$.

(b) In the second experiment, you drop all the $(n+1)$ balls uniformly on $[0,1]$, independent of each other. Let $Y$ denote the number of red balls to the left of the blue ball as before. Find $\mathbb{P}(Y=k)$, for $k=0,\ldots,n$.

**Solution**

(a) Let $U_1,\ldots,U_n \overset{\text{i.i.d.}}{\sim}$ Unif($[0,1]$) be the red ball locations. Then, we have

$$\mathbb{P}(X=k)=\mathbb{E}_{U_1,\ldots,U_n}[\mathbb{P}(X=k|U_1,\ldots,U_n)]=\mathbb{E}[U_{(k+1)}-U_{(k)}]=\frac{k+1}{n+1}-\frac{k}{n+1}=\frac{1}{n+1},$$

where we use the fact that the order statistic $U_{(k)}\sim$ Beta$(k,n-k+1)$. Note that we take $U_{(0)}\equiv 0$ and $U_{(n+1)}\equiv 1$ formally in the above.

(b) Now, let $U_{n+1}\sim$ Unif($[0,1]$) be the location of the blue ball. Then, conditioning on $U_{n+1}$, the other $n$ red balls must be arranged so that $k$ of them fall below $U_{n+1}$ and $n-k$ fall above $U_{n+1}$:

$$\mathbb{P}(Y=k)=\mathbb{E}_{U_{n+1}}\left[\binom{n}{k}U_{n+1}^k\cdot(1-U_{n+1})^{n-k}\middle|U_{n+1}\right]=\binom{n}{k}\cdot\frac{\Gamma(k+1)\cdot\Gamma(n-k+1)}{\Gamma(n+2)}=\frac{1}{n+1},$$

where we recognize the integrand as the kernel of a Beta$(k+1,n-k+1)$ pdf.

**Problem 11** (2019 September, # 1). (i) Suppose that $X$ is a nonnegative random variable. Show that

$$\mathbb{P}(X>t)\le\frac{\mathbb{E}X}{t},\qquad\text{for any } t>0.$$

(ii) Suppose that $Y\sim N(0,1)$. Show that

$$\mathbb{P}(Y>t)\le e^{-t^2/2},\qquad\text{for any } t>0.$$

Hint: you can assume without proof that $\mathbb{E}[e^{\lambda Y}]=e^{\lambda^2/2}$ for all $\lambda\in\mathbb{R}$.

**Solution**

(i) This is Markov's inequality. We have

$$\mathbb{E}[X]\ge\mathbb{E}[X\cdot\mathbf{1}\{X>t\}]\ge t\cdot\mathbb{P}(X>t).$$

(ii) We have for any $\lambda>0$, using part (i) and the hint,

$$\mathbb{P}(Y>t)\le\mathbb{P}(e^{Y\lambda}>e^{\lambda t})\le\frac{\mathbb{E}[e^{Y\lambda}]}{e^{\lambda t}}=\exp(\lambda^2/2-\lambda t).$$

Letting $\lambda=t$, we are done.

**Problem 12** (2019 September, # 2). (i) Let $X$ be a random variable distributed according to an exponential law with expectation $1/\lambda$, where $\lambda$ is a positive constant. Given the real positive random variable $X$, let us define the discrete variable $Y=\lceil X\rceil$, where $\lceil\cdot\rceil$ is the ceiling function, i.e., the function which rounds any real number upwards to the closest integer. For instance, $\lceil 14.3\rceil=15$ and $\lceil 14.8\rceil=15$. Show that the random variable $Y$ follows a geometric distribution and identify the parameter.

(ii) Show that for any continuous random variable $W$ with a strictly increasing cumulative distribution function $F$, we have that $F(W)\sim$ Uniform$[0,1]$.

(iii) Using the results of (i) and (ii), propose an algorithm to simulate a realization of the geometric random variable in (i), from $U\sim$ Uniform($[0,1]$).

**Solution**

(i) We have

$$\mathbb{P}(Y = k) = \mathbb{P}(k - 1 < X \leq k) = e^{-(k-1)\lambda} - e^{-\lambda k} = p^{k-1}(1 - p),$$

for $p = e^{-\lambda}$. Thus, $Y$ is geometric with parameter $p$.

(ii) Since $F$ is strictly increasing, it has an inverse $F^{-1}$. Then, for any $x \in [0, 1]$:

$$\mathbb{P}(F(W) \leq x) = \mathbb{P}(W \leq F^{-1}(x)) = F(F^{-1}(x)) = x.$$

Thus, $F(W) \sim$ Unif$([0, 1])$.

(iii) The exponential cdf $x \mapsto 1 - e^{-\lambda x}$ is strictly increasing on $[0, \infty)$. Thus, we can sample $U \sim$ Unif$([0, 1])$ and then apply the cdf's inverse and round: $\lceil -\log(1 - U)/\lambda \rceil$ which will be distributed as Geometric$(p)$ by (i) and (ii) where we choose $\lambda := \log(1/p)$.

**Problem 13** (2020 May, # 3)**.** Suppose $N, \{X_i\}_{i \geq 1}$ are i.i.d. Poisson random variables with mean $1$. Let $T = \sum_{i=1}^{N} X_i$.

(i) Compute expectation $\mathbb{E}(T)$.

(ii) Compute variance $\mathrm{Var}(T)$.

(iii) Find $\mathbb{P}(T = 1)$ as explicitly as possible.

**Solution**

(i) $\mathbb{E}[\mathbb{E}[T|N = n]] = \mathbb{E}[N] = 1$.

(ii) We have

$$\mathrm{Var}(T) = \mathbb{E}[\mathrm{Var}(T|N)] + \mathrm{Var}(\mathbb{E}[T|N]) = \mathbb{E}[N] + \mathrm{Var}(N) = 2.$$

(iii) We have

$$\begin{aligned}\mathbb{P}(T = 1) = \mathbb{E}[\mathbb{P}(T = 1|N)] = \mathbb{E}\left[\sum_{i=1}^{N} e^{-1} \cdot \left(e^{-1}\right)^{N-1}\right] = \mathbb{E}[Ne^{-N}] = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} \cdot n \cdot e^{-n} = e^{-2} \sum_{n=1}^{\infty} \frac{e^{-(n-1)}}{(n-1)!} \\ = e^{1/e - 2}.\end{aligned}$$

**Problem 14** (2021 May, # 8)**.** Daniel and Ann alternatively toss a fair coin. Daniel tosses the coin first, then Ann tosses the coin, then it is again Daniel's turn and so on. We record the sequence. If there is a head followed by a tail the game ends and the person who tosses the tail wins. What is the probability that Daniel wins the game?

*Hint: Call the event of Daniel winning the game $A$, and let $x = P(A)$. Also, let $B$ denote the event that Daniel sees a Head in the first toss. Use the law of total probability to write down*

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

**Solution**

Following the hint, we have

$$\begin{aligned}P(A) &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= \frac{1}{2}P(A|B) + \frac{1}{2}P(A|B^c) \\ &= \frac{1}{2}P(\text{Daniel wins}|\text{first toss is heads}) + \frac{1}{2}P(\text{Daniel wins}|\text{first toss is tails}).\end{aligned}$$

If the first toss is tails, then Ann cannot win in the second round. By symmetry, we can imagine Ann's second round is actually the first round of the game (i.e., Daniel and Ann switch places) and in this case we see that

$P(\text{Ann wins}|\text{first toss is tails}) = P(\text{Daniel wins}) = P(A)$. Thus, from the above, we conclude

$$P(\text{Daniel wins}|\text{first toss is tails}) = 1 - P(\text{Ann wins}|\text{first toss is tails}) = 1 - P(A).$$

Plugging this into our earlier equation, we have

$$P(A) = \frac{1}{2}P(A|B) + \frac{1}{2}(1 - P(A)).$$

It remains to compute $P(A|B) = P(\text{Daniel wins}|\text{first toss is heads})$. Let $x_1, x_2, \ldots$ be the values of the coin tosses in order, i.e. $x_i \in \{T, H\}$. We first note that

$$1 = P(A|B) + P(A^c|B) = P(\text{Daniel wins}|x_1 = H) + P(\text{Ann wins}|x_1 = H).$$

The second probability on the RHS above is again by law of total probability and conditioning on $x_2$,

$$P(\text{Ann wins}|x_1 = H) = \frac{1}{2}P(\text{Ann wins}|x_1 = x_2 = H) + \frac{1}{2}P(\text{Ann wins}|x_1 = H, x_2 = T) = \frac{1}{2}P(\text{Ann wins}|x_1 = x_2 = H) + \frac{1}{2} \cdot 1.$$

Again, by symmetry, the probability of Ann winning conditional on her first tossing heads is equal to the probability of Daniel winning conditional on his first tossing heads. Thus, $P(\text{Ann wins}|x_1 = x_2 = H) = P(A|B)$. Then, we have

$$1 = P(A|B) + \frac{1}{2}P(A|B) + \frac{1}{2} \implies P(A|B) = \frac{1}{3}.$$

Plugging this into our earlier equation, we have

$$P(A) = 1/6 + 1/2 \cdot (1 - P(A)) \implies P(A) = 4/9.$$

**Problem 15** (2021 Sept, # 3)**.** Let $X$ and $Y$ be two jointly distributed random variables with finite expectations and variances. Show that $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$.

**Solution**

Decompose $(Y - \mathbb{E}[Y])^2$ as $(Y - \mathbb{E}[Y|X] + \mathbb{E}[Y|X] - \mathbb{E}[Y])^2$ and expand the square. The cross-term disappears and then by tower law the identity follows.

## 1.2 Additional Practice

**Problem 16** (Casella & Berger, Exercise 2.7)**.** Let $X$ have pdf $f_X(x) = \frac{2}{9}(x+1)$ for $-1 \le x \le 2$. Find the pdf of $Y = X^2$.

**Solution**

We have

$$\begin{aligned}
\mathbb{P}(Y \le y) &= \mathbb{P}(X^2 \le y) \\
&= \begin{cases} \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) & y \le 1 \\ \mathbb{P}(-1 \le X \le \sqrt{y}) & y > 1. \end{cases} \\
&= \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx & y \le 1 \\ \int_{-1}^{\sqrt{y}} f_X(x)\,dx & y > 1. \end{cases}
\end{aligned}$$

Differentiating this with respect to $y$ gives

$$f_Y(y) = \begin{cases} \frac{2}{9} \cdot \frac{1}{\sqrt{y}} & y \le 1 \\ \frac{1}{9} + \frac{1}{9} \cdot \frac{1}{\sqrt{y}} & y > 1. \end{cases}$$

**Problem 17** (Casella & Berger, Exercise 2.14). Let $X$ be a continuous, nonnegative random variable. Show that

$$\mathbb{E}[X] = \int_0^\infty (1 - F_X(x))\, dx,$$

where $F_X(x)$ is the cdf of $X$.

**Solution**

$$\begin{aligned}\int_0^\infty 1 - F_X(x)\, dx &= \int_0^\infty \int_x^\infty f_X(y)\, dy\, dx \\ &= \int_0^\infty \int_0^y dx \cdot f_X(y)\, dy \\ &= \int_0^\infty y \cdot f_X(y)\, dy \\ &= \mathbb{E}[X].\end{aligned}$$

**Problem 18** (Casella & Berger, Exercise 2.18). Show that if $X$ is a continuous random variable, then

$$\min_a \mathbb{E}|X - a| = \mathbb{E}[X - m|,$$

where $m$ is the median of $X$.

**Solution**

We have $\mathbb{E}|X - a| = \int_{-\infty}^a -(x-a) \cdot f_X(x)\, dx + \int_a^\infty (x-a) \cdot f_X(x)\, dx$. Differentiating this with respect to $a$, we obtain

$$\frac{d}{da}\mathbb{E}|X - a| = \int_{-\infty}^a f(x)\, dx - \int_a^\infty f(x)\, dx.$$

Setting this equal to $0$ we get $a$ is the median of $X$. We can verify this is indeed a minimum since $\frac{d^2}{da^2}\mathbb{E}|X - a| = 2f(a) > 0$.

**Problem 19** (Casella & Berger, Exercise 2.31). Does a distribution exist for which $M_X(t) = \frac{t}{1-t}$ for $|t| < 1$? If yes, find it. If no, prove it.

**Solution**

The answer is no since any mgf must satisfy $M_X(0) = \mathbb{E}[e^0] = 1$ but $\frac{t}{1-t}|_{t=0} = 0$.

**Problem 20** (Casella & Berger, Exercise 3.18). Let $Y$ be a negative binomial random variable with parameters $r$ and $p$, where $p$ is the success probability. Show that as $p \to \infty$, the mgf of the random variable $pY$ converges to that of a gamma distribution with parameters $r$ and $1$ ($r$ is the shape parameter). This is known as *convergence in distribution*, which we will discuss in Review Session 5.

**Solution**

The mgf of $Y$ is $M_Y(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$ and so the mgf of $pY$ is $M_{pY}(t) = \left(\frac{p}{1-(1-p)e^{pt}}\right)^r$. Then, by L'Hopital's rule, we have

$$\lim_{p \to 0} \frac{p}{1 - (1-p)e^{pt}} = \lim_{p \to 0} \frac{1}{(p-1)te^{pt} + e^{pt}} = \frac{1}{1-t},$$

so that $M_{pY}(t) \overset{p \to 0}{\to} (1-t)^{-r}$, which is the mgff of a $\Gamma(r, 1)$ distribution.

**Problem 21** (Casella & Berger, Exercise 3.44). For any random variable $X$ for which $\mathbb{E}[X^2]$ and $\mathbb{E}[|X|]$ exist, show that $\mathbb{P}(|X| \geq b)$ does not exceed either $\mathbb{E}[X^2]/b^2$ or $\mathbb{E}[|X|]/b$, where $b$ is a positive constant. If $X$ has pdf $f_X(x) = e^{-x}$ for $x > 0$ show that one bound is better when $b = 3$ and the other when $b = \sqrt{2}$.

**Solution**

By Markov, we have $\mathbb{P}(|X| \geq b) \leq \mathbb{E}[|X|]/b$. By applying Markov to $X^2$ (also called *Chebyshev's inequality*), we have $\mathbb{P}(|X| \geq b) = \mathbb{P}(X^2 \geq b^2) \leq \mathbb{E}[X^2]/b^2$.
For $X \sim \exp(1)$, $\mathbb{E}[|X|] = \mathbb{E}[X] = 1$ and $\mathbb{E}[X^2] = 2$. For $b = 3$, the first bound is thus $\mathbb{E}[|X|]/b = 1/3$ while the second bound is $\mathbb{E}[X^2]/b^2 = 2/9$. Thus, the second moment bound is better. Meanwhile, for $b = \sqrt{2}$, $\mathbb{E}[|X|]/b = 1/\sqrt{2} < 1 = \mathbb{E}[X^2]/b^2$.