

Review Session 5 – Solutions

1 Solutions

1.1 Previous Core Competency Problems

Problem 1 (2018 Summer Practice, # 11). Suppose that $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\lambda/n)$.

- (a) What is the distribution of $\sum_{i=1}^n X_i$.
- (b) Compute $\lim_{n \rightarrow \infty} \mathbb{P}(\sum_{i=1}^n X_i = k)$, where k is any fixed nonnegative integer, and hence show that $\sum_{i=1}^n X_i$ converges in distribution to a random variable Y .
- (c) Compute $\mathbb{E}[Y(Y-1)]$, where Y is as in part (b).

Solution

(a) $\sum_{i=1}^n X_i \sim \text{Binomial}(n, \lambda/n)$.

(b) We claim $Y = \sum_{i=1}^n X_i \xrightarrow{d} \text{Poisson}(\lambda)$. Indeed, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^n X_i = k\right) = \lim_{n \rightarrow \infty} \frac{n(n-1) \cdots (n-k+1)}{k! n^k} \cdot \frac{\lambda^k}{(1-\lambda/n)^k} (1-\lambda/n)^n = \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} (1-\lambda/n)^n = \frac{\lambda^k}{k!} e^{-\lambda}.$$

(c) $\mathbb{E}[Y(Y-1)] = \mathbb{E}[Y^2] - \mathbb{E}[Y] = \lambda + \lambda^2 - \lambda = \lambda^2$.

Problem 2 (2018 Summer Practice, # 16). Farmers in the Hudson Valley pack apples into bags of approximately 10 pounds, but due to the variation in apples the actual weight varies. We may model the weight of a bag as uniformly distributed in $[9.5, 10.5]$ and independent of other bags. The farmers load 1200 bags onto a truck with maximal admissible load of 13000 pounds. Find a simple approximation to the probability that the truck is overloaded, expressed in terms of the Normal distribution.

Solution

Let W_i be the weight of the i -th bag, so that $\mathbb{E}[W_i] = 10$ and $\text{Var}(W_i) = 1/12$. Then, by CLT:

$$\frac{\bar{W}_{1200} - 10}{1/\sqrt{12}} \sqrt{1200} \xrightarrow{d} \mathcal{N}(0, 1).$$

Thus,

$$\mathbb{P}\left(\sum_{i=1}^{1200} W_i > 13000\right) = \mathbb{P}\left(\frac{\bar{W} - 10}{1/\sqrt{12}} \sqrt{1200} > 100\right) \approx \mathbb{P}(\mathcal{N}(0, 1) > 100)$$

Problem 3 (2018 Summer Practice, # 19). suppose for every $n \geq 1$, A_n is a real symmetric matrix of size $n \times n$, whose eigenvalues $(\lambda_1, \dots, \lambda_n)$ satisfies the following properties:

- (i) $\max_{i=1}^n |\lambda_i| \xrightarrow{n \rightarrow \infty} 0$.
- (ii) $\sum_{i=1}^n \lambda_i^2 = 1$.

Find the asymptotic distribution of $\sum_{i,j=1}^n A_n(i, j) X_i X_j$, where $\{X_i\}_{i \geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$.

Solution

By the eigendecomposition of A_n , it suffices to compute the limiting distribution of $S_n := \sum_{i=1}^n X_i^2 \lambda_i$. We will proceed via Lyapunov CLT for the triangular array $\{X_i \lambda_i\}_{i,n}$. Condition (ii) gives us that the normalization constant is $\sum_{i=1}^n \lambda_i^2 = 1$ for all $n \in \mathbb{N}$. Next, we verify the third-moment condition:

$$\lim_n \sum_{i=1}^n \mathbb{E}[|X_i^2 \lambda_i - \lambda_i|^3] = \lim_n \sum_{i=1}^n |\lambda_i|^3 \mathbb{E}[|X^2 - 1|^3] \leq (\max_i |\lambda_i|) \cdot \mathbb{E}[|X^2 - 1|^3] \xrightarrow{n \rightarrow \infty} 0$$

Thus, by Lyapunov CLT, $S_n - \sum_{i=1}^n \lambda_i \xrightarrow{d} \mathcal{N}(0, \text{Var}(X^2))$.

Problem 4 (2018 September, # 3). Suppose that, for $n \geq 1$, X_n is a random variable taking values in $\{1/n, 2/n, \dots, n/n\}$ with equal probability $1/n$.

- (i) Show that X_n converges in distribution, as $n \rightarrow \infty$? What is its weak limit?
- (ii) Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined as $f(x) = x \sin(x)$, for $x \in [0, 1]$. Using the above or otherwise, show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

Solution

- (i) We will show this by an mgf computation. We have the mgf of X_n is

$$\sum_{k=1}^n \frac{1}{n} \cdot e^{t \cdot k/n} = \begin{cases} \frac{1}{n} \left(\frac{e^{t \cdot \left(\frac{n+1}{n}\right)} - 1}{e^{t/n} - 1} \right) & t \neq 0 \\ 1 & t = 0 \end{cases}.$$

Next, we take the limit as $n \rightarrow \infty$. Suppose $t \neq 0$. We have $\lim_n e^{t \cdot \left(\frac{n+1}{n}\right)} - 1 = e^t - 1$. Next, via L'Hôpital's rule:

$$\lim_n n(e^{t/n} - 1) = \lim_n \frac{e^{t/n} - 1}{1/n} = \lim_n \frac{-\frac{t}{n^2} e^{t/n}}{-1/n^2} = \lim_n t e^{t/n} = t.$$

Thus, the mgf of X_n goes to $(e^t - 1)/t$ for $t \neq 0$ and 1 for $t = 0$. This is precisely the mgf of a $\text{Unif}([0, 1])$, whence $X_n \xrightarrow{d} \text{Unif}([0, 1])$.

- (ii) f is a bounded, continuous function on $[0, 1]$. Thus, by portmanteau theorem,

$$\lim_n \mathbb{E}[f(X_n)] = \mathbb{E}_{X \sim \text{Unif}([0,1])}[f(X)],$$

which is the desired result.

Remark 1.1. (ii) also follows from just the definition of the Riemann integral $\int_0^1 f(x) dx$ (which of course exists for continuous f).

Problem 5 (2018 September, # 7). Suppose X_1, \dots, X_n are i.i.d. with $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$. Define

$$Y_i := \prod_{j=1}^i X_j, \quad \text{for } i = 1, \dots, n.$$

- (i) Find the joint distribution of (Y_1, Y_2) .
- (ii) Derive the limiting distribution of $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i$.

Solution

Working out the various cases, we see that (Y_1, Y_2) takes values $(\pm 1, \pm 1)$ uniformly. We have each Y_i is ± 1 with probability $1/2$ and, by induction, the $\{Y_i\}_{i=1}^n$'s are mutually independent since $\mathbb{P}(Y_{n+1}|Y_1, \dots, Y_n) = \mathbb{P}(Y_{n+1})$. Thus, by CLT, $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \xrightarrow{d} \mathcal{N}(0, 1)$.

Problem 6 (2019, May # 2). Let Z_1, \dots, Z_n be i.i.d. random variables with density f . Suppose that (i) $\mathbb{P}(Z_i > 0) = 1$, and (ii) f is continuous on $[0, \epsilon)$, for some $\epsilon > 0$. Let $\lambda := f(0)$. Let

$$X_n = n \min\{Z_1, \dots, Z_n\}.$$

Show that X_n converges in distribution, and find the limiting distribution.

Solution

We have for $c > 0$:

$$\mathbb{P}(X_n > c) = \prod_{i=1}^n \mathbb{P}(Z_i > c/n) = (1 - F(c/n))^n.$$

Next, we consider $\lim_n \log((1 - F(c/n))^n) = \lim_n n \log(1 - F(c/n))$. By L'Hôpital's rule:

$$\lim_n \frac{\log(1 - F(c/n))}{1/n} = \lim_{x \rightarrow \infty} \frac{\frac{f(c/x) \cdot c/x^2}{1 - F(c/x)}}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{-f(c/x) \cdot c}{1 - F(c/x)}.$$

Since f is continuous near 0, so is its antiderivative with $\lim_{x \rightarrow \infty} F(c/x) = 0$. Thus, the above RHS is equal to $-f(0) \cdot c = -\lambda \cdot c$, meaning $\mathbb{P}(X_n > c) \rightarrow e^{-\lambda \cdot c}$. Thus, $X_n \xrightarrow{d} \exp(\lambda)$.

Problem 7 (2019 May, # 8). Suppose you have a quadratic form $\mathbf{X}_n^T A_n \mathbf{X}_n$, where $\mathbf{X}_n \sim N_n(\mathbf{0}_{n \times 1}, \mathbf{I}_{n \times n})$, and A_n is a symmetric $n \times n$ matrix with 0 on the diagonal. Let $(\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{n,n})$ denote the eigenvalues of A_n , and let $\|\lambda\|_{2,n} := \sqrt{\sum_{i=1}^n \lambda_{i,n}^2}$ denote the ℓ_2 -norm of the eigenvalues.

(a) If $\lim_{n \rightarrow \infty} \frac{\max_{i=1, \dots, n} |\lambda_{i,n}|}{\|\lambda\|_{2,n}} = 0$, show that $T_n := \frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \xrightarrow{d} N(0, 1)$ as $n \rightarrow \infty$.

[Hint: You may use Lyapunov's¹ CLT. Note that the trace of a square matrix is the sum of its eigenvalues.]

(b) If

$$\lim_{n \rightarrow \infty} \frac{\lambda_{1,n}}{\|\lambda\|_{2,n}} = 1,$$

show that $T_n \xrightarrow{d} \chi_1^2 - 1$.

Solution

(a) By an eigendecomposition of A_n , we can write

$$\frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \stackrel{d}{=} \frac{1}{\|\lambda\|_{2,n}} \sum_{i=1}^n X_{i,n}^2 \cdot \lambda_{i,n}.$$

Next, since $\text{Tr}(A_n) = 0 = \sum_{i=1}^n \lambda_{i,n}$, we have $\mathbb{E} \left[\frac{1}{\|\lambda\|_{2,n}} \mathbf{X}_n^T A_n \mathbf{X}_n \right] = 0$. Finally, for the sake of using Lyapunov's CLT, we last need the third moment condition which amounts to bounding:

$$\lim_n \frac{1}{\|\lambda\|_{2,n}^3} \sum_{i=1}^n |\lambda_{i,n}|^3 \leq \left(\frac{\max_i |\lambda_{i,n}|}{\|\lambda\|_{2,n}} \right) \cdot \frac{\sum_{i=1}^n \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} \xrightarrow{n \rightarrow \infty} 0.$$

¹**Lyapunov's CLT:** Suppose that $\{Z_1, Z_2, \dots\}$ is a sequence of independent random variables such that Z_i has finite expected value μ_i and variance σ_i^2 . Define $s_n^2 := \sum_{i=1}^n \sigma_i^2$. If $\lim_{n \rightarrow \infty} \frac{1}{s_n^3} \sum_{i=1}^n \mathbb{E}[|Z_i - \mu_i|^3] = 0$ is satisfied, then $\frac{1}{s_n} \sum_{i=1}^n (Z_i - \mu_i) \xrightarrow{d} N(0, 1)$.

Thus, by Lyapunov CLT $T_n \xrightarrow{d} \mathcal{N}(0, \text{Var}(X_{1,1}^2)) = \mathcal{N}(0, 2)$.

(b) First, we write

$$T_n = X_{n,1}^2 \cdot \frac{\lambda_1}{\|\lambda\|_{2,n}} + \sum_{i=2}^n \frac{X_{n,i}^2 \cdot \lambda_i}{\|\lambda\|_{2,n}}.$$

The first term on the RHS goes to χ_1^2 in distribution by Slutsky. It suffices to show the second term on the RHS goes to -1 . In fact, we claim this will follow from:

$$\frac{1}{\|\lambda\|_{2,n}} \left(\sum_{i=2}^n X_{n,i}^2 \cdot \lambda_i - \mathbb{E}[X_{n,i}^2] \cdot \lambda_i \right) \xrightarrow{P} 0. \quad (1)$$

To verify (1), we have for any fixed $\epsilon > 0$, by Chebyshev:

$$\begin{aligned} \mathbb{P} \left(\frac{1}{\|\lambda\|_{2,n}} \left| \sum_{i=2}^n X_{n,i}^2 \cdot \lambda_{i,n} - \mathbb{E}[X_{n,i}^2] \cdot \lambda_{i,n} \right| \geq \epsilon \right) &\leq \frac{\text{Var} \left(\sum_{i=2}^n X_{n,i}^2 \cdot \lambda_{i,n} \right)}{\|\lambda\|_{2,n}^2 \cdot \epsilon^2} \\ &= \frac{1}{\|\lambda\|_{2,n}^2 \cdot \epsilon^2} \sum_{i=2}^n \text{Var}(X_{n,i}^2) \cdot \lambda_{i,n}^2. \end{aligned}$$

Then, it suffices to show $\frac{\sum_{i=2}^n \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} \rightarrow 0$. But, this follows immediately from realizing

$$1 = \lim_n \frac{\lambda_{1,n}^2}{\|\lambda\|_{2,n}^2} + \frac{\sum_{i=2}^n \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2} = 1 + \lim_n \frac{\sum_{i=2}^n \lambda_{i,n}^2}{\|\lambda\|_{2,n}^2}.$$

Finally, we have that $\sum_{i=1}^n \lambda_{i,n} = 0$ for all $n \in \mathbb{N}$ gives us

$$\frac{1}{\|\lambda\|_{2,n}} \sum_{i=2}^n \mathbb{E}[X_{n,i}^2] \cdot \lambda_{i,n} = \sum_{i=2}^n \frac{\lambda_{i,n}}{\|\lambda\|_{2,n}} \rightarrow -1.$$

Putting everything together, we conclude $T_n \xrightarrow{d} \chi_1^2 - 1$.

Remark 1.2. Note that our use of Chebyshev to establish (1) can be considered a strong version of Law of Large Numbers for triangular arrays (e.g., see Theorem 2.2.4 in Durrett's *Probability: Theory and Examples*).

Problem 8 (2019 May, # 9). Let $Y_n = \prod_{i=1}^n X_i$ where X_1, \dots, X_n are i.i.d. nonnegative non-degenerate random variables with mean $\mathbb{E}(X_i) = 1$. Prove that $Y_n \xrightarrow{P} 0$ as $n \rightarrow \infty$ when: (i) $\mathbb{P}(X_1 = 0) > 0$, and (ii) $\mathbb{P}(X_1 = 0) = 0$.

Solution

(i) For $\epsilon > 0$, we have

$$\mathbb{P}(Y_n > \epsilon) \leq \mathbb{P}(\forall i \in [n] : X_i > 0) = \mathbb{P}(X_1 > 0)^n.$$

However, since $\mathbb{P}(X_1 > 0) < 1$, the RHS above goes to 0 as $n \rightarrow \infty$. Thus, $Y_n \xrightarrow{P} 0$.

(ii) We may now assume WLOG that $X_i > 0$ everywhere. Consider the transformation $X \mapsto \log(X)$. We have, by Jensen:

$$\mathbb{E}[\log(X_1)] < \log \mathbb{E}[X_1] = 0.$$

Thus, SLLN gives $\log(Y_n) = \sum_{i=1}^n \log(X_i) \xrightarrow{a.s.} -\infty$, meaning $Y_n \xrightarrow{P} 0$.

Problem 9 (2019 May, # 10). Let $f_{X,Y}(x, y)$ be a bivariate density and let $(X_1, Y_1), \dots, (X_N, Y_N)$ be i.i.d. $f_{X,Y}$. Let $w(\cdot)$ be an

arbitrary probability density function. Let

$$\hat{f}_X(x) = \frac{1}{N} \sum_{i=1}^N \frac{f_{X,Y}(x, Y_i) w(X_i)}{f_{X,Y}(X_i, Y_i)}.$$

Show that, for any $x \in \mathbb{R}$, $\hat{f}_X(x) \xrightarrow{P} f_X(x)$, where f_X is the marginal density of X_1 .

Solution

It suffices to show

$$\mathbb{E}_{X_1, Y_1} \left[\frac{f_{X,Y}(x, Y_1) \cdot w(X_1)}{f_{X,Y}(X_1, Y_1)} \right] = f_X(x),$$

whence the result will follow from LLN. Indeed, we have

$$\begin{aligned} \mathbb{E}_{X_1, Y_1} \left[\frac{f_{X,Y}(x, Y_1) \cdot w(X_1)}{f_{X,Y}(X_1, Y_1)} \right] &= \int f_{X,Y}(x, Y_1) \cdot w(X_1) d(X_1, Y_1) \\ &= \int f_{X,Y}(x, Y_1) \int w(X_1) dX_1 dY_1 \\ &= \int f_{X,Y}(x, Y_1) dY_1 \\ &= f_X(x), \end{aligned}$$

where choosing the order of integration is justified by Tonelli's theorem.

Problem 10 (2019 September, # 6). Suppose that X_1, X_2, \dots are i.i.d. having an exponential distribution with mean 1. Show that

$$\frac{\max_{1 \leq k \leq n} X_k}{\log n} \xrightarrow{P} 1 \text{ as } n \rightarrow \infty$$

where \xrightarrow{P} denotes convergence in probability.

Solution

We have for $\epsilon \in (0, 1)$:

$$\begin{aligned} \mathbb{P} \left(\left| \frac{\max_{1 \leq k \leq n} X_k}{\log(n)} - 1 \right| > \epsilon \right) &= \mathbb{P} \left(\max_{1 \leq k \leq n} X_k \geq (1 + \epsilon) \log(n) \right) + \mathbb{P} \left(\max_{1 \leq k \leq n} X_k \leq (1 - \epsilon) \cdot \log(n) \right) \\ &\leq n \cdot e^{-(1+\epsilon) \log(n)} + (1 - e^{-(1-\epsilon) \cdot \log(n)})^n \\ &= \frac{n}{n^{\epsilon+1}} + \left(1 - \frac{1}{n^{1-\epsilon}} \right)^n \end{aligned}$$

However, both of these last terms go to 0 as $n \rightarrow \infty$ (the second limit can be computed with l'Hôpital's rule). Thus, we've shown the definition of $\max_{1 \leq k \leq n} X_k / \log(n) \xrightarrow{P} 1$.

Problem 11 (2020 May, # 2). Let X_1, X_2, \dots, X_n denote n independent and identically distributed observations from $\text{Uniform}(0, 1)$. We order these observations according to their distance from $x = 0.75$ and call the ordered ones $X_{(1)}^x, X_{(2)}^x, \dots, X_{(n)}^x$. Note that $X_{(1)}^x$ and $X_{(n)}^x$ are, respectively, the closest and farthest observations from $x = 0.75$.

- (i) Prove that $X_{(1)}^x$ converges to 0.75 in probability.
- (ii) What does $X_{(n)}^x$ converge to in probability? Prove your answer.

Solution

(i)

$$\mathbb{P}(|X_{(1)}^x - 0.75| > \epsilon) = \prod_{i=1}^n \mathbb{P}(|X_i - 0.75| > \epsilon) = \prod_{i=1}^n (1 - 2\epsilon) \xrightarrow{n \rightarrow \infty} 0.$$

(ii) We claim $X_{(n)}^x \xrightarrow{P} 0$. Indeed, for $\epsilon < 0.75$, we have $X_{(n)}^x > \epsilon \implies \forall i \in [n] : X_i > \epsilon$ so that

$$\mathbb{P}(|X_{(n)}^x| > \epsilon) \leq \prod_{i=1}^n \mathbb{P}(X_i > \epsilon) = (1 - \epsilon)^n \xrightarrow{n \rightarrow \infty} 0$$

Problem 12 (2020 September, # 2). Suppose that X_1, \dots, X_{2n} are i.i.d. $U[0, 1]$. Let $Y_i = X_{2i-1} + X_{2i}$ for $1 \leq i \leq n$.

(a) Find the limiting distribution of Y_1 .

(b) Find the limiting distribution of $\sqrt{n}(2 - Y_{(n)})$ as $n \rightarrow \infty$.

Solution

(a) Y_1 has cdf

$$\mathbb{P}(X_1 + X_2 \leq t) = \begin{cases} \int_0^t \mathbb{P}(X_1 \leq t - x) dx = \int_0^t (t - x) dx = t^2/2 & t \in [0, 1] \\ \int_{t-1}^1 \mathbb{P}(X_1 \leq t - x) dx + \int_0^{t-1} 1 dx = 2t - t^2/2 - 1 & t \in [1, 2] \end{cases}.$$

(b) For $w > 0$ and n large enough we have, using part (a),

$$\begin{aligned} \mathbb{P}(\sqrt{n}(2 - Y_{(n)}) > w) &= \mathbb{P}\left(Y_{(n)} < 2 - \frac{w}{\sqrt{n}}\right) \\ &= \left(2\left(2 - \frac{w}{\sqrt{n}}\right) - \frac{1}{2}\left(2 - \frac{w}{\sqrt{n}}\right)^2 - 1\right)^n \\ &= \left(1 - \frac{w^2}{2n}\right)^n. \end{aligned}$$

This last expression goes to $e^{-w^2/2}$ as $n \rightarrow \infty$. Thus, the limiting distribution of $\sqrt{n}(2 - Y_{(n)})$ has cdf $F(w) = 1 - e^{-w^2/2}$ for $w > 0$.

Problem 13 (2021 September, # 5). Suppose $\{\xi_i\}_{i \geq 0}$ are i.i.d. $\mathcal{N}(0, 1)$ random variables. Find the constant c such that

$$\frac{\max_{1 \leq i \leq n} X_i}{\sqrt{\log(n)}} \xrightarrow{P} c,$$

for each of the following three cases where $\{X_i\}_{i \geq 1}$ is defined.

(i) $X_i = \xi_i$ for $i \geq 1$.

(ii) $X_i = \xi_i + \xi_0$ for $i \geq 1$.

(iii) $X_i = \frac{\xi_i + \xi_{i-1}}{\sqrt{2}}$ for $i \geq 1$.

Solution

(i) We first write the cdf of $\max_i \xi_i / \sqrt{\log(n)}$. This is

$$\mathbb{P}\left(\frac{\max_i \xi_i}{\sqrt{\log(n)}} \leq x\right) = \mathbb{P}(\xi_0 \leq x\sqrt{\log(n)})^n = \Phi(x\sqrt{\log(n)})^n,$$

where $\Phi(\cdot)$ is the standard normal cdf. First, if $x \leq 0$, then we see that

$$\lim_n \Phi(x\sqrt{\log(n)})^n \leq \lim_n (1/2)^n = 0.$$

So, it suffices to compute $\lim_n \Phi(x\sqrt{\log(n)})^n$ for $x > 0$. We apply L'Hopital's rule to $\log(\Phi(x\sqrt{\log(n)})^n)$ (let $\phi(\cdot)$ be the standard normal pdf) giving:

$$\begin{aligned} \lim_n \frac{\log(\Phi(x\sqrt{\log(n)}))}{1/n} &= \lim_n \frac{\frac{1}{\Phi(x\sqrt{\log(n)})} \cdot \phi(x\sqrt{\log(n)}) \cdot \frac{x}{\sqrt{\log(n)}} \cdot \left(\frac{1}{2}\right) \cdot \frac{1}{n}}{-1/n^2} \\ &= \lim_n -\frac{n^{1-x^2/2}}{2\sqrt{\log(n)}} \cdot \frac{x}{\Phi(x\sqrt{\log(n)})} \\ &= \begin{cases} -\infty & x \in (0, \sqrt{2}) \\ 0 & x \geq \sqrt{2} \end{cases} \end{aligned}$$

Thus, $\lim_n \Phi(x\sqrt{\log(n)}) \rightarrow \mathbf{1}\{x \geq \sqrt{2}\}$ which is the cdf of the constant $\sqrt{2}$. Thus, $c = \sqrt{2}$ for (i).

(ii) $\frac{\max_i \xi_i + \xi_0}{\sqrt{\log(n)}} = \frac{\xi_0}{\sqrt{\log(n)}} + \frac{\max_i \xi_i}{\sqrt{\log(n)}} \xrightarrow{P} \sqrt{2}$ by (i) and the fact that $\xi_0/\sqrt{\log(n)} \xrightarrow{P} 0$.

(iii) We have

$$\begin{aligned} \mathbb{P}\left(\max_i X_i \leq x\sqrt{\log(n)}\right) &= \mathbb{P}\left(-\max_i X_i \geq -x\sqrt{\log(n)}\right) \\ &= \mathbb{P}\left(\min_i X_i \geq -x\sqrt{\log(n)}\right). \end{aligned}$$

where we replace each X_i with $-X_i$ by symmetry at the last step. Next, if $\min_i X_i \geq -x\sqrt{\log(n)}$, then for each $i \in [n]$, $\frac{\xi_i + \xi_{i-1}}{\sqrt{2}} \geq -x\sqrt{\log(n)}$ which means that for each $i \in [n]$ either $\xi_i \geq -x\sqrt{\log(n)}/2$ or $\xi_{i-1} \geq -x\sqrt{\log(n)}/2$. In particular, this is true for every odd $i \in [n]$. Thus, we can bound the RHS probability above by:

$$\begin{aligned} \mathbb{P}\left(i \in 1, \dots, \lfloor n/2 \rfloor : \max(\xi_{2i}, \xi_{2i+1}) \geq -x\sqrt{\log(n)}/2\right) &\leq \left(2 \cdot \mathbb{P}\left(\mathcal{N}(0, 1) \geq -x\sqrt{\log(n)}/2\right)\right)^{\lfloor n/2 \rfloor} \\ &\leq (2 - 2\Phi(-x\sqrt{\log(n)}/2))^{\lfloor n/2 \rfloor}. \end{aligned}$$

We again apply L'Hopital to find the limit of this last expression. To get rid of the annoying $\lfloor n/2 \rfloor$ we'll replace the exponent $\lfloor n/2 \rfloor$ with the smaller $n/4$ which only makes the above bound larger (and as we shall see will be suitable for our end results). We have

$$\begin{aligned} \lim_n \log((2 - 2\Phi(-x\sqrt{\log(n)}/2))^{n/4}) &= \lim_n \frac{\log(2 - 2\Phi(-x\sqrt{\log(n)}/2))}{4/n} \\ &= \lim_n \frac{(2 - 2\Phi(-x\sqrt{\log(n)}/2))^{-1} \cdot (-2\phi(-x\sqrt{\log(n)}/2)) \cdot \left(-\frac{x/\sqrt{2}}{\sqrt{\log(n)}/2}\right) \cdot \frac{1}{2n}}{-4/n^2} \\ &\propto -\frac{n^{1-x^2/4} \cdot x}{\sqrt{\log(n)}} \cdot \frac{1}{1 - \Phi(-x\sqrt{\log(n)}/2)}. \end{aligned}$$

This last expression goes to $-\infty$ for $x \in [0, 2)$ and goes to 0 for $x = 2$. Thus, taking $\exp(\cdot)$, we find that

$$\lim_n \mathbb{P}\left(\max_i X_i \leq x\sqrt{\log(n)}\right) \leq \begin{cases} 0 & x \in [0, 2) \\ 1 & x = 2 \end{cases}.$$

On the other hand, $\max_i \frac{\xi_i}{\sqrt{2}} \leq \frac{x}{2} \sqrt{\log(n)} \implies \max_i X_i \leq x \sqrt{\log(n)}$ so that

$$\mathbb{P}\left(\max_i X_i \leq x \sqrt{\log(n)}\right) \geq \mathbb{P}\left(\max_i \xi_i \leq x \sqrt{\log(n)/2}\right) = \mathbb{P}\left(\frac{\max_i \xi_i}{\sqrt{\log(n)}} \leq x/\sqrt{2}\right) \xrightarrow{n \rightarrow \infty} \mathbf{1}\{x \geq 2\},$$

where the last part follows from part (i). Putting these upper and lower bounds on the cdf of $\max_i X_i / \sqrt{\log(n)}$ together, we see that $\frac{\max_i X_i}{\sqrt{2 \log(n)}} \xrightarrow{P} 2$.

Problem 14 (2021 September, # 7). Let $X_1, X_2, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ (F denotes the CDF). Our goal is to estimate $\gamma = F(0) + 2F(1)$. We employ the following estimate

$$\hat{\gamma} = \frac{1}{n} \left(\sum_{i=1}^n \mathbf{1}\{X_i \leq 0\} + 2 \sum_{i=1}^n \mathbf{1}\{X_i \leq 1\} \right),$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function.

- (i) Calculate $\mathbb{E}[\hat{\gamma}]$.
- (ii) What is the limiting distribution of $\sqrt{n}(\hat{\gamma} - \gamma)$? Justify your answer.

Solution

(i) $\mathbb{E}[\hat{\gamma}] = F(0) + 2F(1) = \gamma$.

(ii) By CLT, $\sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\mathbf{1}\{X \leq 0\} + 2\mathbf{1}\{X \leq 1\}))$. This variance is

$$\mathbb{E}[(\mathbf{1}\{X \leq 0\} + 2\mathbf{1}\{X \leq 1\})^2] - \gamma^2 = 5F(0) + 4F(1) - (F(0) + 2F(1))^2.$$

Problem 15 (2021 September, # 8). Answer the following questions.

- (i) Suppose that $(X_n, Y_n) \xrightarrow{d} \mathcal{N}(0, \Sigma)$ in distribution with $\Sigma = [2, 1; 1, 1]$. What does $(X_n - Y_n)^2$ converge in distribution? Prove your answer.
- (ii) Suppose that $(X_n, \sqrt{n}Y_n) \xrightarrow{d} \mathcal{N}(0, \Sigma)$ in distribution with $\Sigma = [2, 1; 1, 1]$. What does $(X_n - Y_n)^2$ converge to in distribution? Prove your answer.
- (iii) Let $X_n \xrightarrow{P} 1$. For each X_n , we pick Y_n uniformly at random from the interval $[0, X_n]$. What does Y_n converge to in distribution? Prove your answer.

Solution

- (i) Let $(Z_1, Z_2) \sim \mathcal{N}(\mathbf{0}_2, \Sigma)$. Then, by continuous mapping theorem, $X_n - Y_n \xrightarrow{d} Z_1 - Z_2 \sim \mathcal{N}(0, 1)$ by Cramer-Wold (where $\text{Var}(Z_1 - Z_2) = 2 + 1 - 2 = 1$). Thus, again by continuous mapping theorem, $(X_n - Y_n)^2 \xrightarrow{d} \chi_1^2$.
- (ii) By Cramer-Wold, we have $\sqrt{n} \cdot Y_n \xrightarrow{d} \mathcal{N}(0, 1)$ which implies $Y_n \xrightarrow{d} 0$ by Slutsky. Then, again by Slutsky, $X_n - Y_n \xrightarrow{d} \mathcal{N}(0, 2)$ so that $(X_n - Y_n)^2 \xrightarrow{d} 2 \cdot \chi_1^2$.
- (iii) I'll assume $X_n > 0$ a.s. for all n or else the problem doesn't make sense. We claim $Y_n \xrightarrow{d} \text{Unif}([0, 1])$. Fix $y \in [0, 1]$ and let $G(x) = \frac{y \cdot \mathbf{1}\{y \in [0, x]\}}{x}$ which is the cdf of $Y_n | X_n = x$ at y . Then, note that $G(y)$ is a bounded and a.s. continuous function. Thus, by Portmanteau theorem, $X_n \xrightarrow{d} 1$ implies $\mathbb{E}[G(X_n)] \rightarrow G(1)$. But, $G(1)$ is just the $\text{Unif}([0, 1])$ cdf evaluated at y . Thus, we've shown convergence of cdfs $\mathbb{E}[G(X_n)] = F_{Y_n}(y) \rightarrow y \cdot \mathbf{1}\{y \in [0, 1]\}$ for all $y \in [0, 1]$ meaning $Y_n \xrightarrow{d} \text{Unif}([0, 1])$.